Wavelets & Turbulent Flow

Some Aspects of Data Analysis

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Motivation & scope

turbulent flow is:

- multi-scale, bearing coherent structures
- inherently 3D
- described by vector quantities
- in general statistically inhomogeneous

wavelet bases:

- allow to decompose a signal w.r.t. space & scale simultaneously

towards:

- wavelet bases suiting the demands of turbulence
- significant quantities derived from the raw coefficients
- feasible computation

Outline

I Generalities of wavelet transform

II Wavelets for the interval

III Multi-dimensional approaches

IV Vector-valued/divergence-free bases

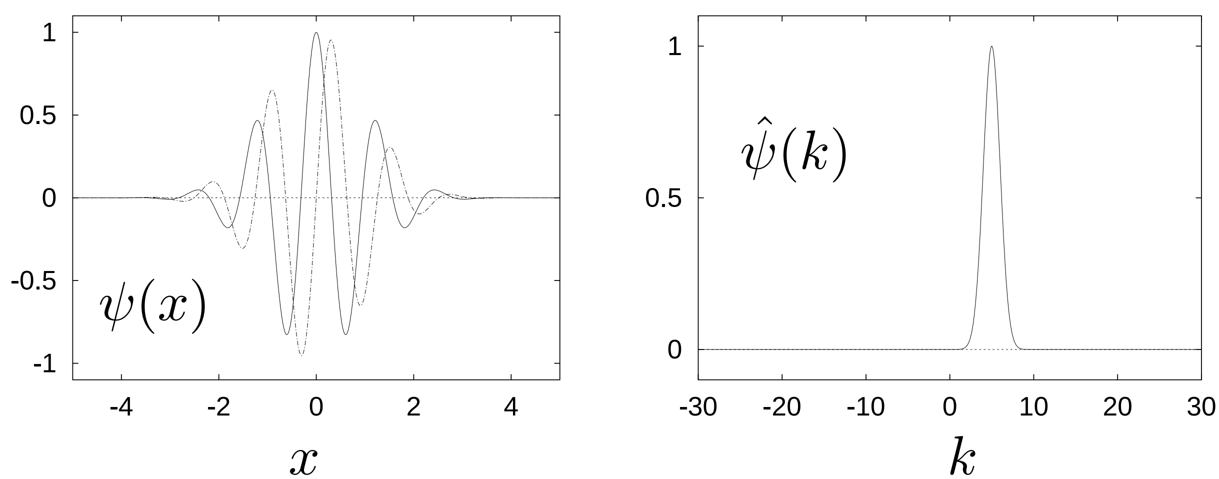
V Implementation issues

What is a wavelet ?

admissibility: function with zero-mean

localization: in physical & frequency space

e.g. Morlet wavelet $\psi(x) = \mathrm{e}^{inx} \cdot \mathrm{e}^{-x^2/2}$



- complex-valued
- barely admissible
- exponential decay

Continuous wavelet transform

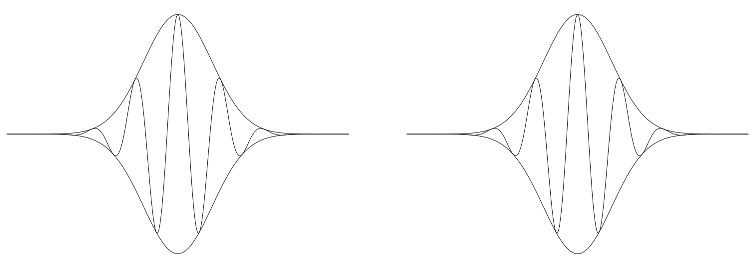
scalar product

$$\tilde{f}(a,b) = \int_x \psi^* \left(\frac{x-b}{a} \right) f(x) \, \mathsf{d}x$$

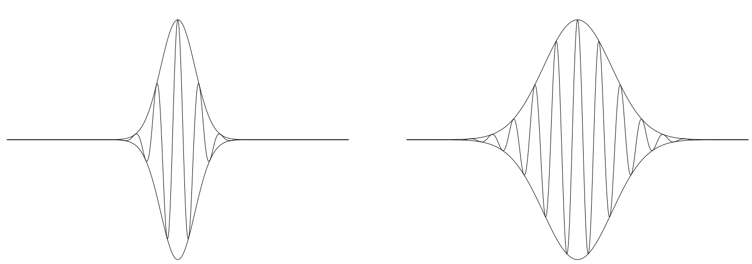
- scale factor $a > 0$
- translation $b \in \mathbb{R}$

wavelet windowed Fourier

$a = 1$



$a = 0.5$



Discrete wavelet transform

- wavelet translated & dilated dyadically

$$\psi_{ji}(x) = 2^{j/2}\,\psi\left(2^j x - i\right) \qquad j, i \in \mathbb{Z}$$

orthogonality:

$$< \psi_{ji}, \psi_{kl} > \equiv \int_x \psi_{ji}\psi_{kl}\mathrm{d}x = \delta_{jk}\delta_{il}$$

- definition of a scaling function

$$\int_x \varphi(x)\mathrm{d}x = 1\,, \quad \varphi_{ji} = 2^{j/2}\,\varphi\left(2^j x - i\right)$$

orthogonality to wavelets & own translates:

$$\begin{aligned} < \varphi_{ji}, \psi_{kl} > &= 0\,, \quad k > j \\ < \varphi_{ji}, \varphi_{jl} > &= \delta_{il} \end{aligned}$$

- various families ψ, φ exist

Multi-resolution analysis

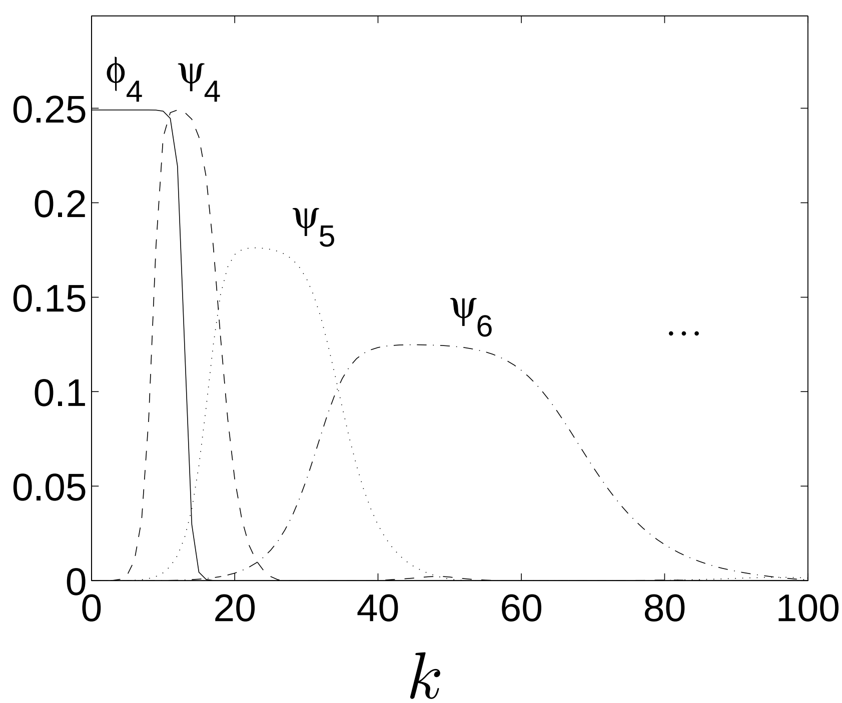
- scaling fcts. generate nested subspaces:

$$
\begin{aligned}
V_j &= \overline{\text{span}}\{\varphi_{ji}\}_{i\in\mathbb{Z}} \\
V_0 \subset V_1 &\subset \ldots V_j \subset V_{j+1} \subset \ldots
\end{aligned}
$$

- wavelet fcts. are complement spaces:

$$
\begin{aligned}
W_j &= \overline{\text{span}}\{\psi_{ji}\}_{i\in\mathbb{Z}} \\
V_{j+1} &= V_j \oplus W_j
\end{aligned}
$$

low-pass/band-pass filters:



Multi-resolution analysis (cont'd.)

particularly: $$L^2(\mathbb{R}) = V_J \oplus \left(\bigoplus_{j \geq J} W_j \right)$$

$\Rightarrow$ decomposition of a signal:

$$\begin{aligned} f(x) &= \sum_{i \in \mathbb{Z}} \overbrace{< f, \varphi_{Ji} >}^{c_{Ji}} \varphi_{Ji} \\ &+ \sum_{j \geq J} \sum_{i \in \mathbb{Z}} \underbrace{< f, \psi_{ji} >}_{d_{ji}} \psi_{ji} \end{aligned}$$

- energy (Parseval)

$$E \equiv \int f(x)^2 \, \mathrm{d}x = \sum_i c_{Ji}^2 + \sum_j \sum_i d_{ji}^2$$

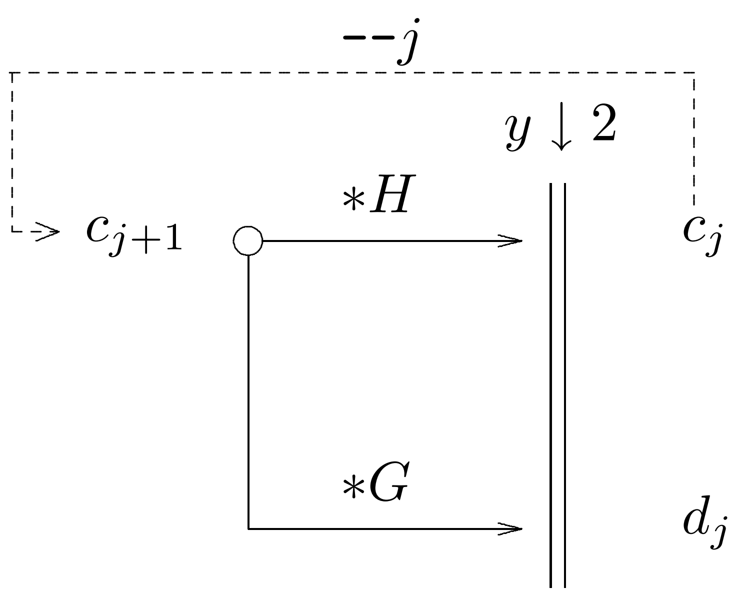
- N to N transform

Mallat algorithm

- identify a low-pass filter related to φ, a band-pass filter from ψ:

$$\varphi\,,\psi \longrightarrow H\,,G$$

- algorithm: from small to large scales



- operation count $\mathcal{O}(N\log_2(N))$

Choice of basis functions φ, ψ

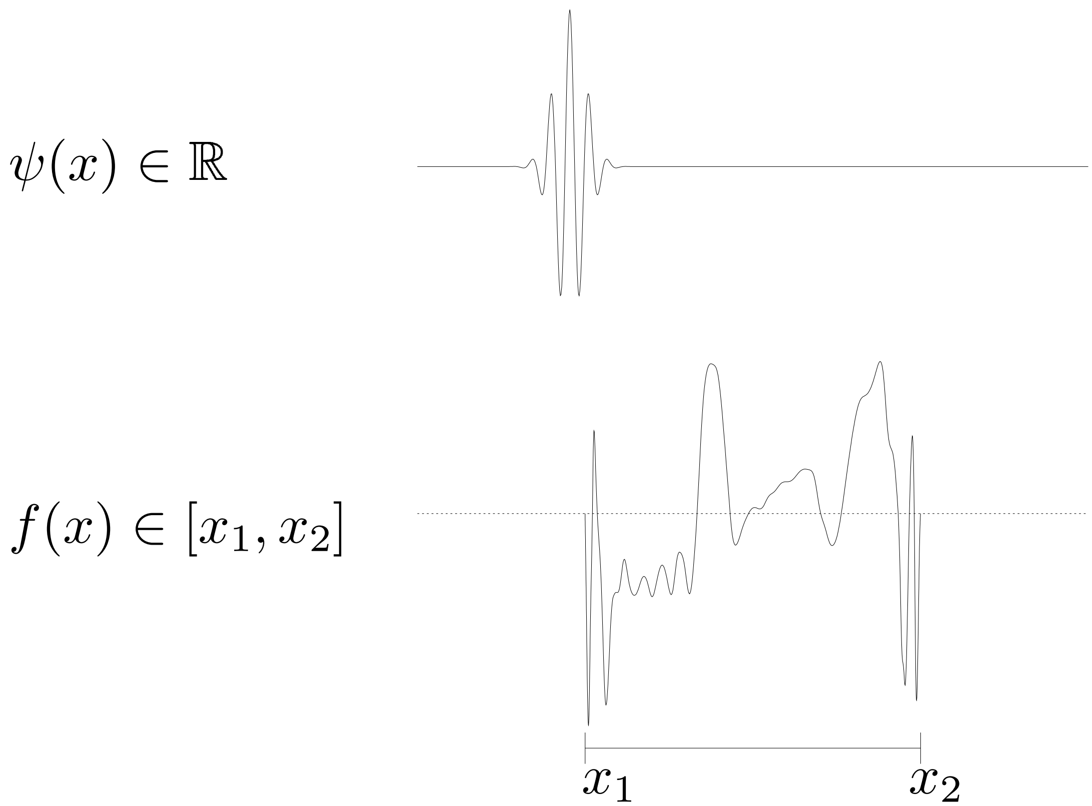
- discrete-orthogonal vs. continuous

  (orthogonality w.r.t. weight unity)

- real-valued vs. complex
- localization (space & scale)
- symmetry
- vanishing moments
- non-compact vs. compact

Bounded signal: problem

- Fourier techniques do not work
- artifacts at boundary:



- boundary region might be particularly interesting (e.g. wall-bounded flow)

$\Rightarrow$ need orthogonal basis on $[x_1, x_2]$

$\circ$ attempts with $\cos$-mapping failed

Polynomial wavelets

Fourier

$$\psi_{ji}(x) = \sum_k a_{ji}(k)\mathrm{e}^{Ikx} \qquad x \in \mathbb{R}/\mathbb{Z}$$

Orthopoly

$$\psi_{ji}(x) = \sum_k a_{ji}(k)P_k(x) \quad x \in [-1,1]$$

with $\int P_k\, P_l\, w(x)\, \mathrm{d}x = \delta_{kl}$

Fischer & Prestin (1997):

- use repro. kernel poly. $K_n = \sum_{k=0}^{n} P_k(x)P_n(y)$

$\varphi_{ni} = K_n(x, y_i^{(n+1)})$, $\psi_{ni} = K_{2n}(x, y_i^{(n)}) - K_n(x, y_i^{(n)})$

- setting $P_k = U_k$ (Chebyshev II) yields:

$$\begin{aligned} <\varphi_{ji}, \varphi_{jl}>_w &= \delta_{il} \\ <\psi_{ji}, \psi_{ml}>_w &= \delta_{il}\,\delta_{jm} \\ <\varphi_{ji}, \psi_{ml}>_w &= 0 \qquad\qquad (m \geq j) \end{aligned}$$

- however: $w(x) = \sqrt{1-x^2}$

Legendre wavelets

- $a_{ji}(k)$ to fulfill orthogonality of $\psi_{ji}(x)$
- choice of P_k determines the weight $w(x)$

$\longrightarrow$ set instead $P_k = L_k$ (Legendre)

(Prestin, pers. comm. 2001)

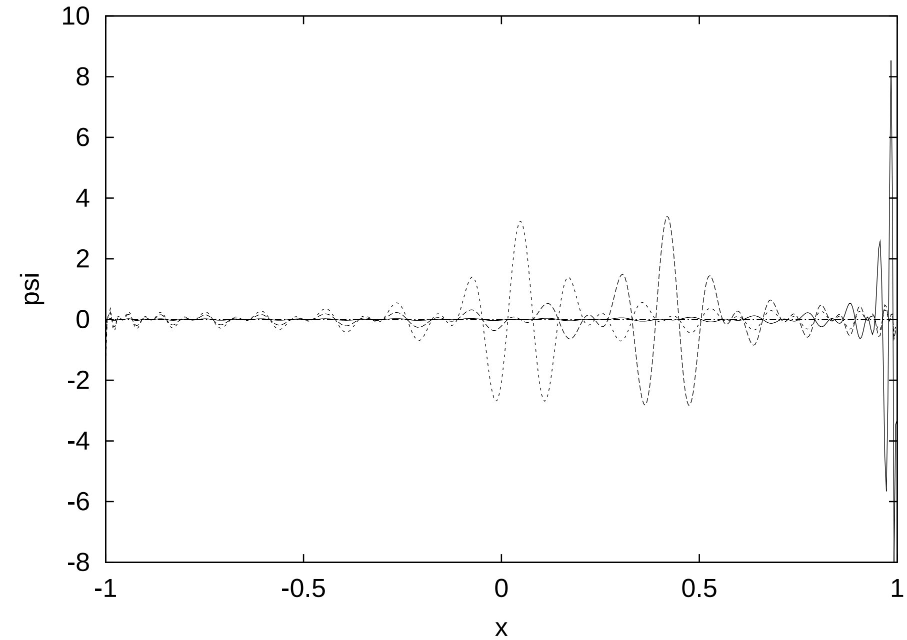
$\Rightarrow w(x) = 1$ as desired

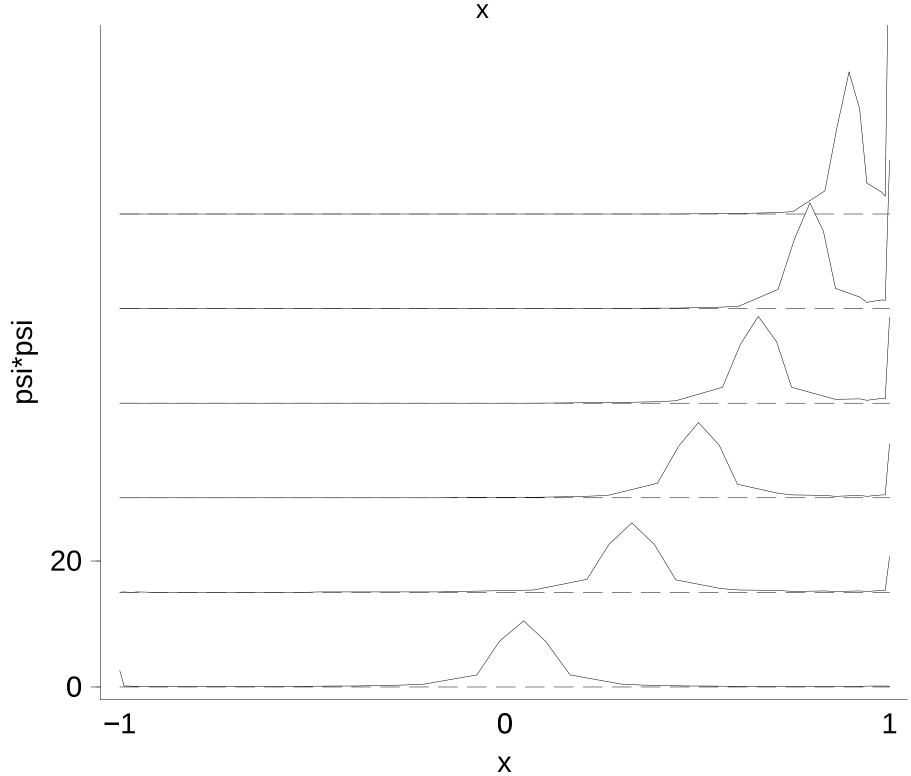
$$\varphi_{ji}(x) = C_{ij}^{\varphi} \sum_{k=0}^{2^j} U_k(y_i^{(2^j+1)}) \cdot \sqrt{k+1/2} \cdot L_k(x) \qquad i = 0 \ldots 2^j$$

$$\psi_{ji}(x) = C_{ij}^{\psi} \sum_{k=2^j+1}^{2^{j+1}} U_k(y_i^{(2^j)}) \cdot \sqrt{k+1/2} \cdot L_k(x) \qquad i = 0 \ldots 2^j - 1$$

with zeroes $y_i^{(n)} = -\cos\left(\frac{(i+1)\,\pi}{n+1}\right)$

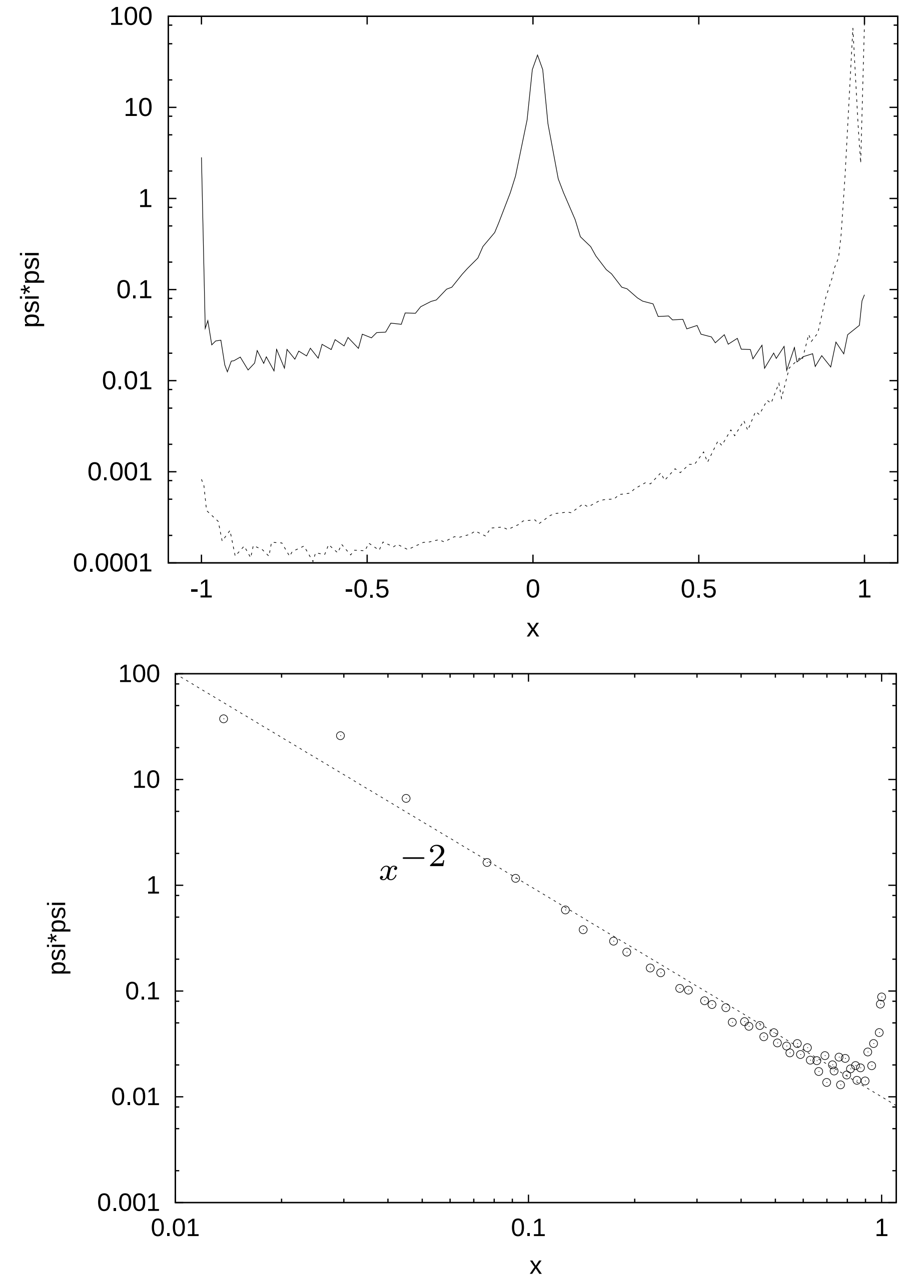
Legendre wavelets – shape





- no strict translational invariance

Legendre wavelets – decay



scale index $j = 7$, position $i = 64, 117$

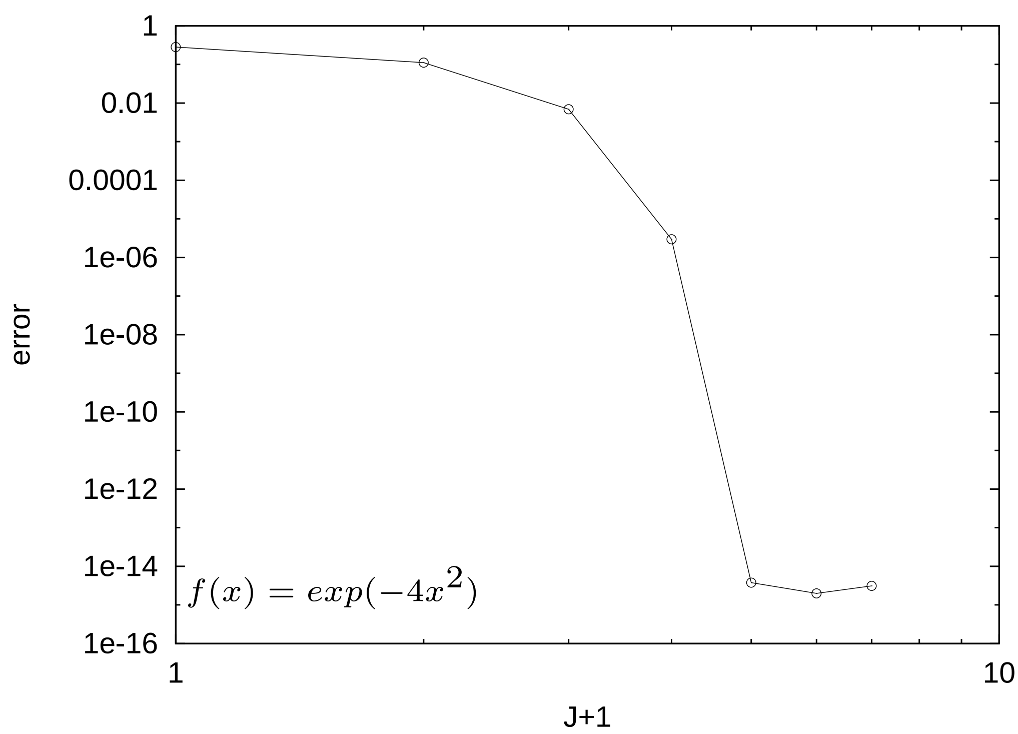
- “energy” contained in tails is small

Legendre wavelets – MRA

- propose (non-classical) MRA, $L_2([-1,1])$

$$
\begin{aligned}
f(x_l) &= c_{00}\varphi_{00}(x_l) + c_{01}\varphi_{01}(x_l) \\
&+ \sum_{j=0}^{J} \sum_{i=0}^{2^j-1} d_{ji}\psi_{ji}(x_l)
\end{aligned}
$$

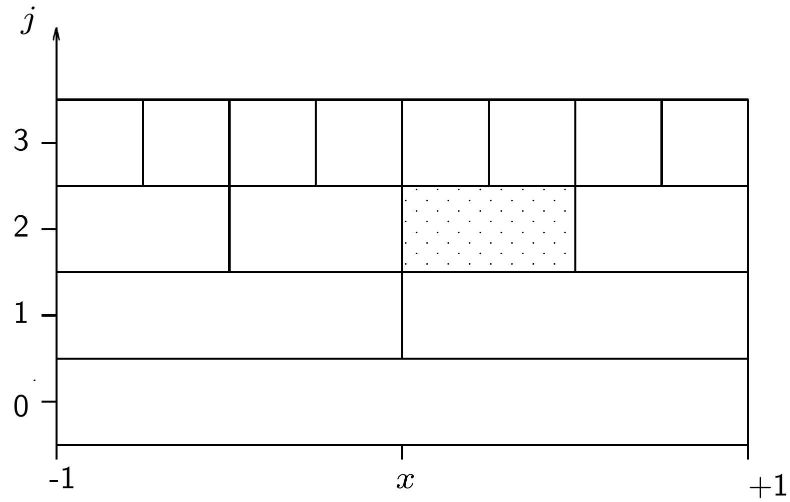
- numerical convergence of the approximation



- energy: $\int f(x)^2 \mathrm{d}x = c_{00}^2 + c_{01}^2 + \sum_{j,i} d_{ji}^2$

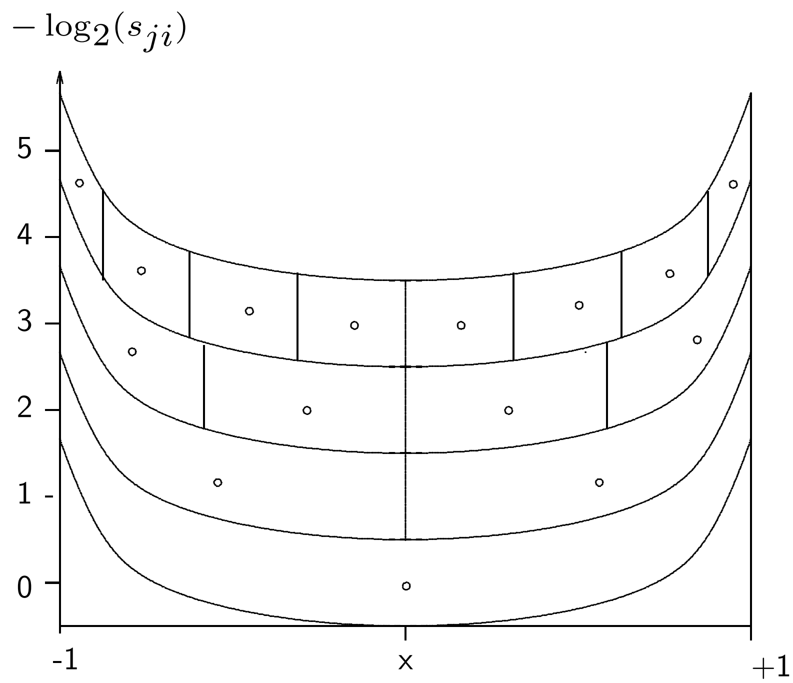
Visualization: scalogram

classical: scale $\sim 2^{-j}$

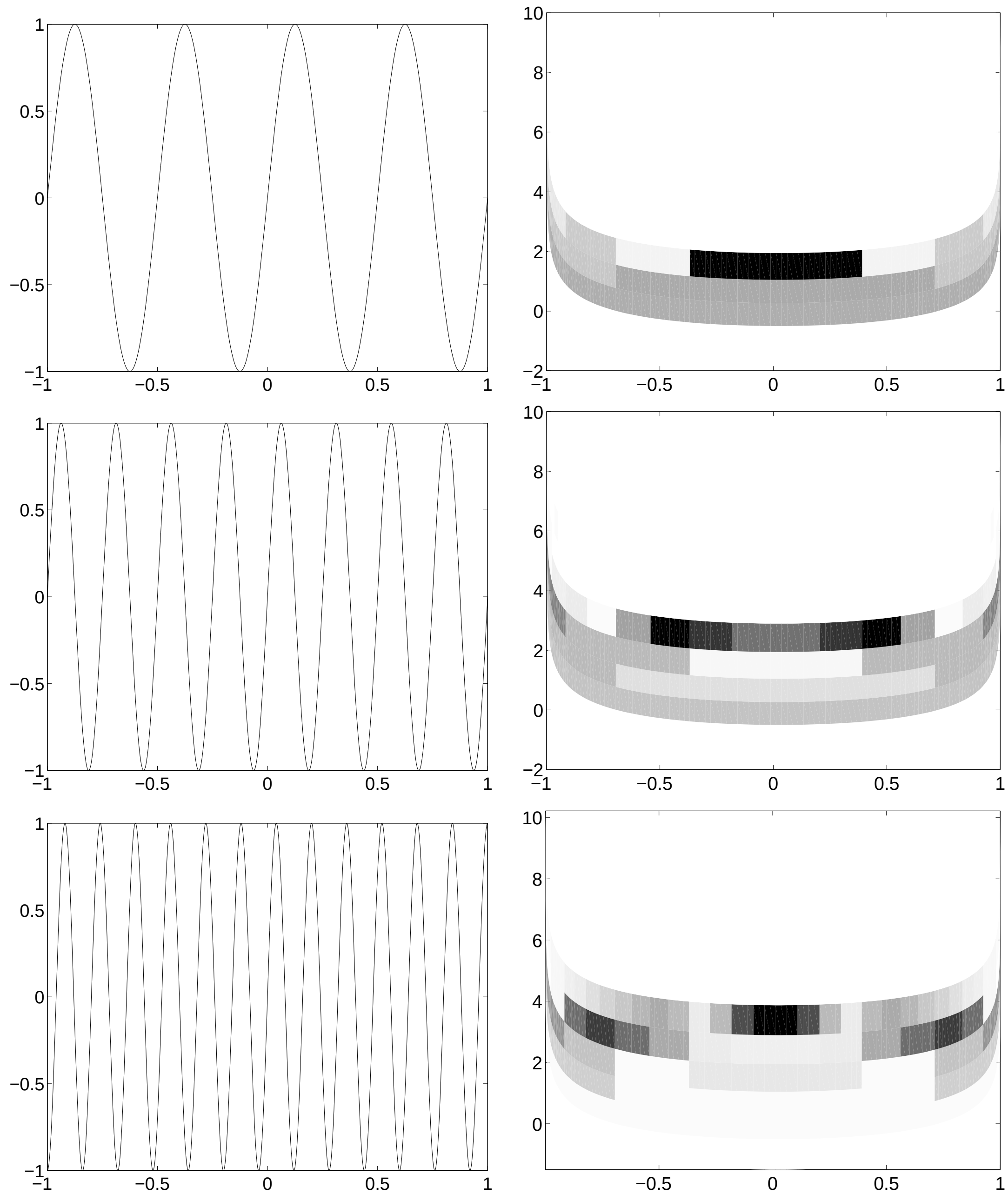


here: centers are on cos-stretched grid

$$\text{scale } s_{ji} = \sin\left(\frac{(i+1)\pi}{2^j+1}\right)\sin\left(\frac{\pi/2}{2^j+1}\right)$$

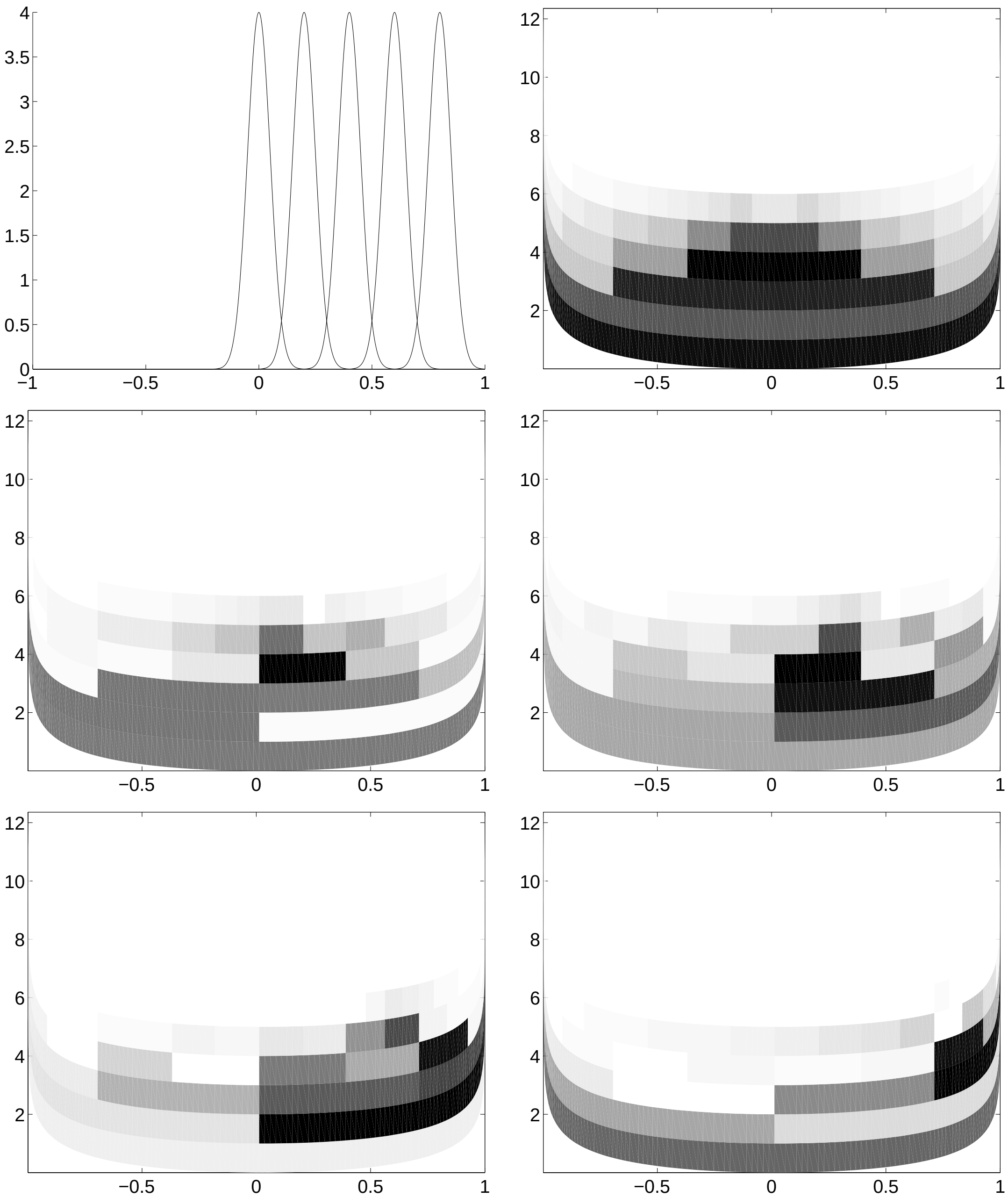


Artificial signal: sine



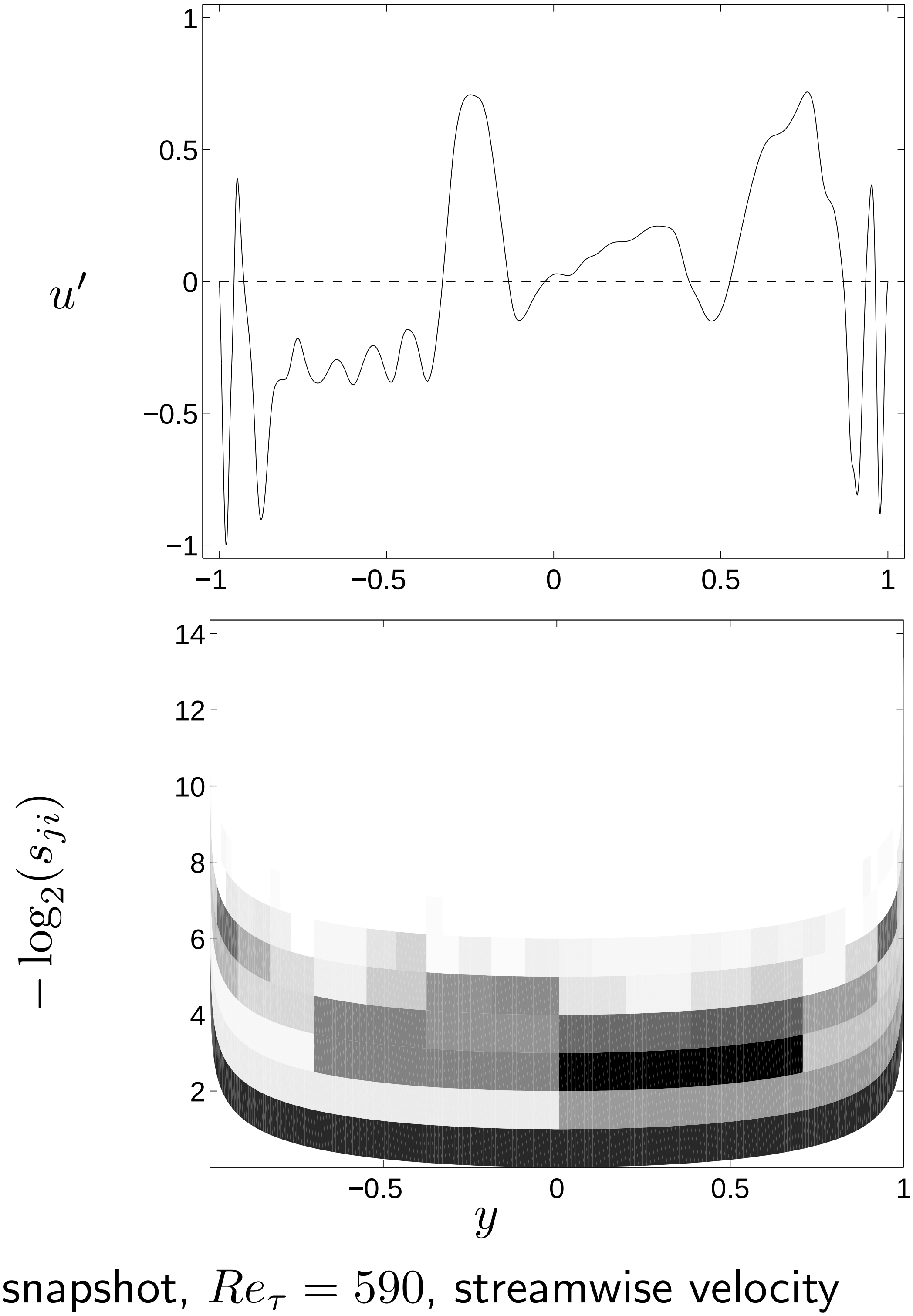
• reading of scale, even if non-periodic

Artificial signal: bump



- scale & localization of bump

Real signal: plane channel flow



- snapshot, $Re_\tau = 590$, streamwise velocity fluctuations, wall-to-wall

Local energy spectrum

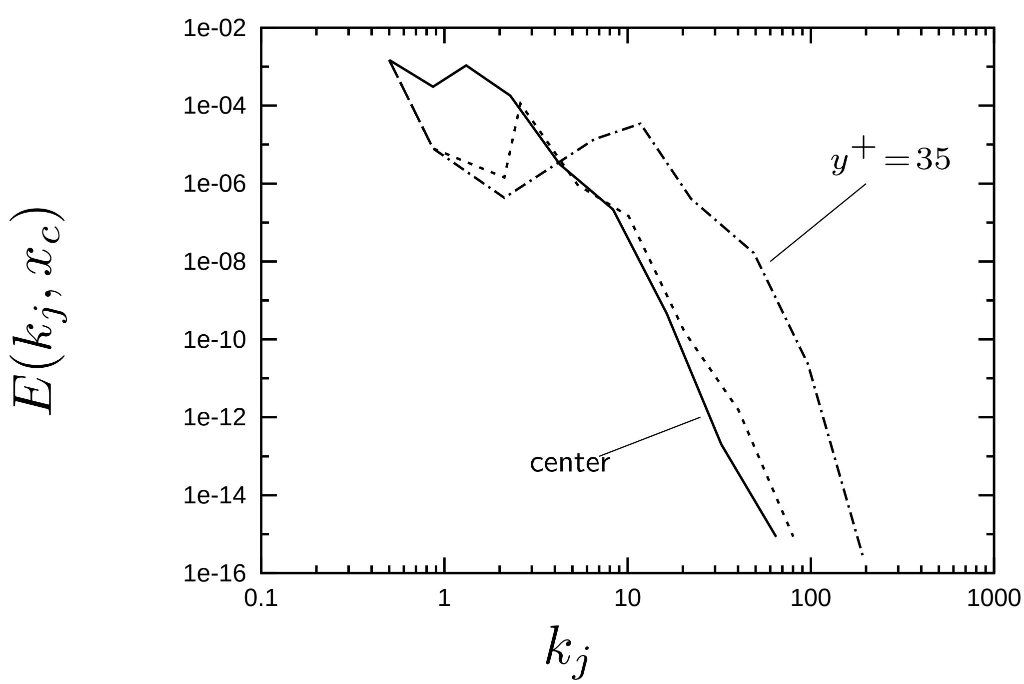
- define power spectral density

$$E(k_j, x_c) \equiv \frac{d_{ji_c}^2}{\Delta k_j} \quad \text{(closest-to-}x_c\text{ index gives } i_c\text{)}$$

and wavenumber or "scalenumber"

$$k_j = \frac{1}{s_{ji_c}}$$

previous channel data:



Future tasks

- improve localization properties in physical space (presently: decay $\sim x^{-1}$)
- conceive a faster algorithm for the transform (presently: $\mathcal{O}(N^2)$)

Multi-D: General approaches

(A) genuinely multi-D basis

$$\psi^{j,\alpha}_{i_x,i_y}(\vec{x})$$

$\boxed{\text{(B)}}$ tensor product of 1D bases

$$\psi^{j_x,j_y}_{i_x,i_y}(x,y) = \tilde{\psi}_{j_x,i_x}(x) \cdot \psi_{j_y,i_y}(y)$$

- scales are "scrambled"

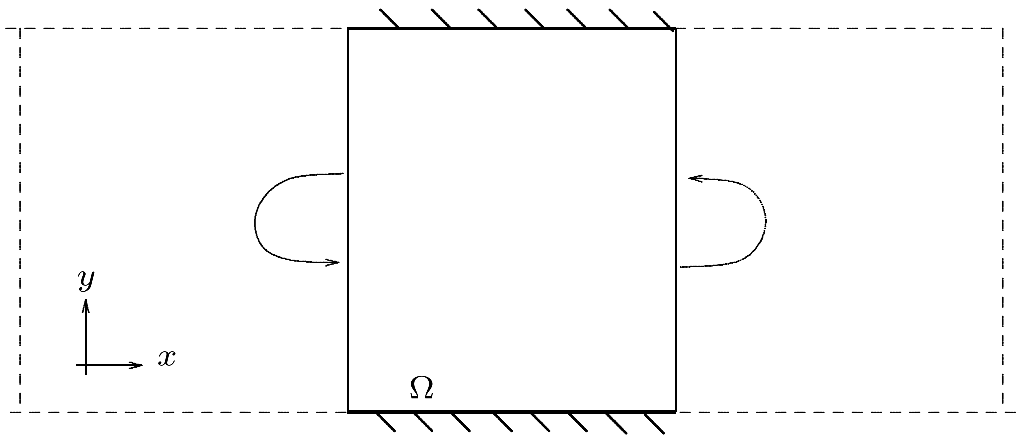
$\boxed{\text{(C)}}$ tensor product of 1D MRA's

$$\begin{aligned}
\psi^{j,1}_{i_x,i_y}(x,y) &= \tilde{\varphi}_{j,i_x}(x) \cdot \psi_{j,i_y}(y) \\
\psi^{j,2}_{i_x,i_y}(x,y) &= \tilde{\psi}_{j,i_x}(x) \cdot \varphi_{j,i_y}(y) \\
\psi^{j,3}_{i_x,i_y}(x,y) &= \tilde{\psi}_{j,i_x}(x) \cdot \psi_{j,i_y}(y)
\end{aligned}$$

- scale index j controls refinement in both directions

2D example: hybrid MRA

$$\Omega = \mathbb{R}/\mathbb{Z} \times [-1, +1]$$

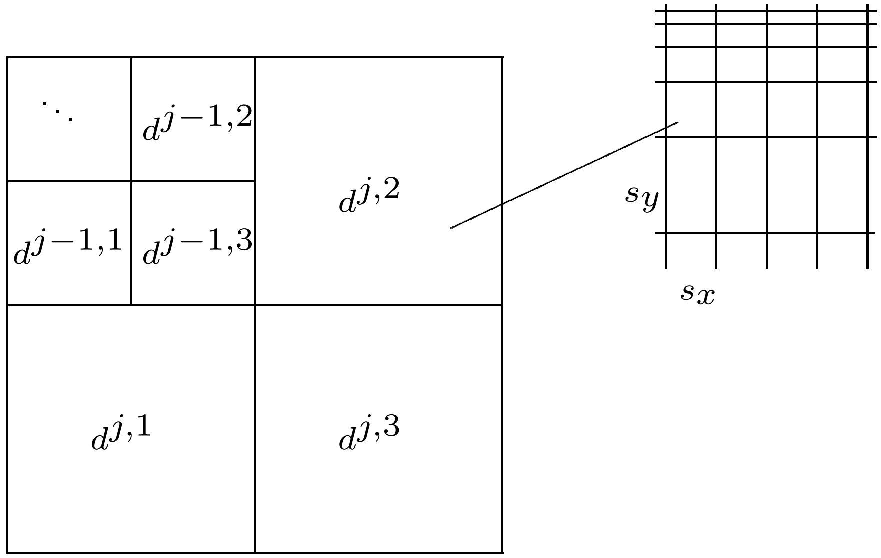


$\underline{x}$: periodic spline wavelets
(Perrier & Basdevant 1989)

$\underline{y}$: Legendre wavelets

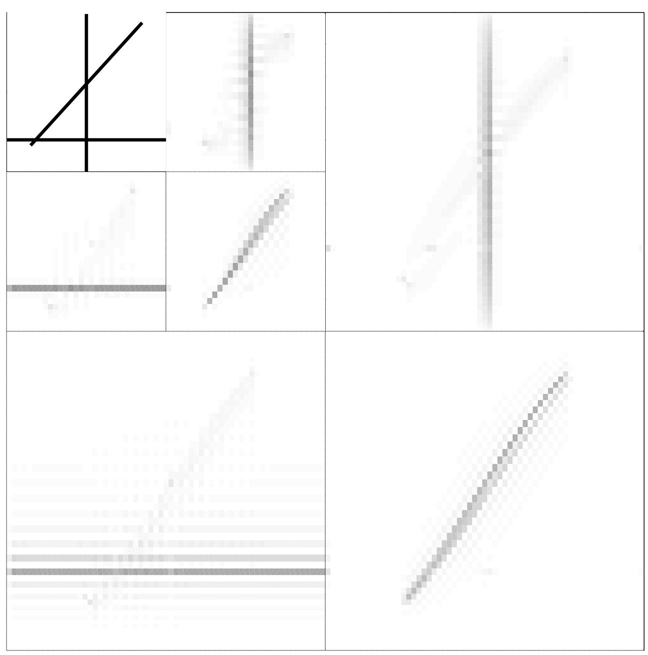
- localization properties differ in x, y
- scale is function of position in y: $s_y(j, i_y)$

2D coefficient scheme (C)



bitmap example:

signal



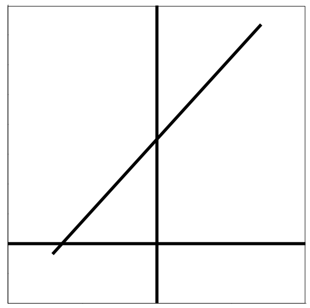
vertical features

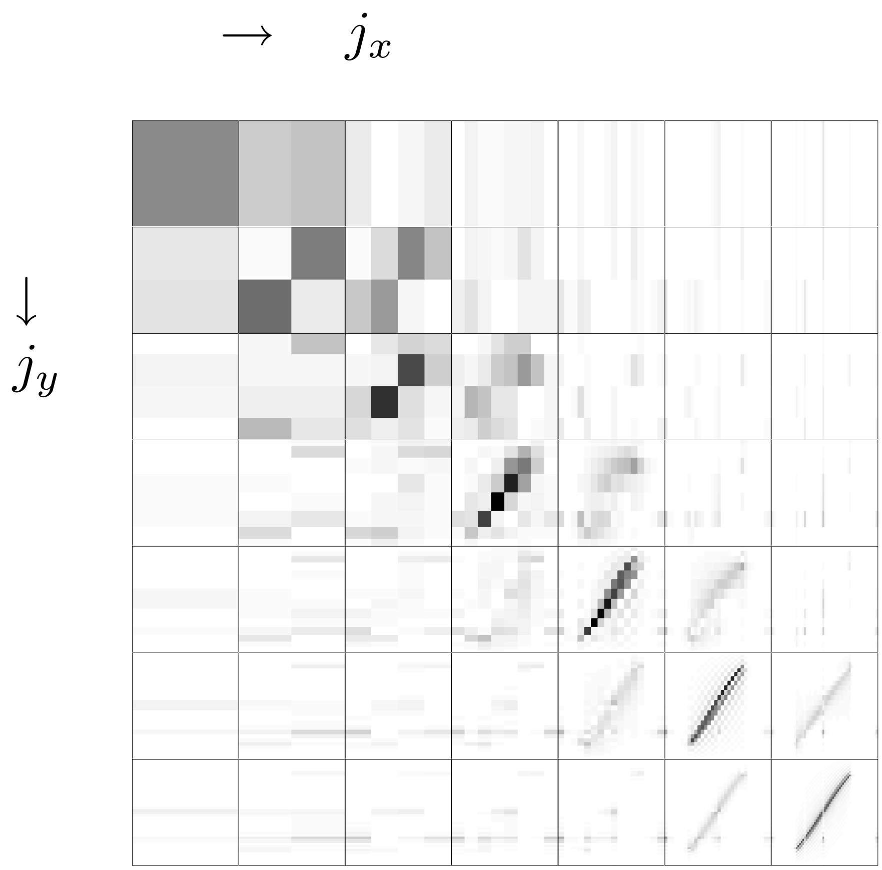
diagonal features

horizontal features

2D coefficient scheme (B)

(previous) bitmap example:

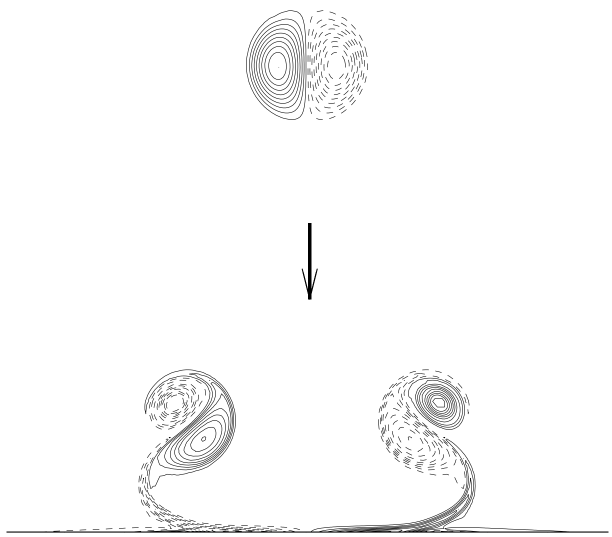




- select scales separately (aspect ratio)

Vortex dipole rebound

$$Re = \frac{u_\infty R}{\nu} = 771$$

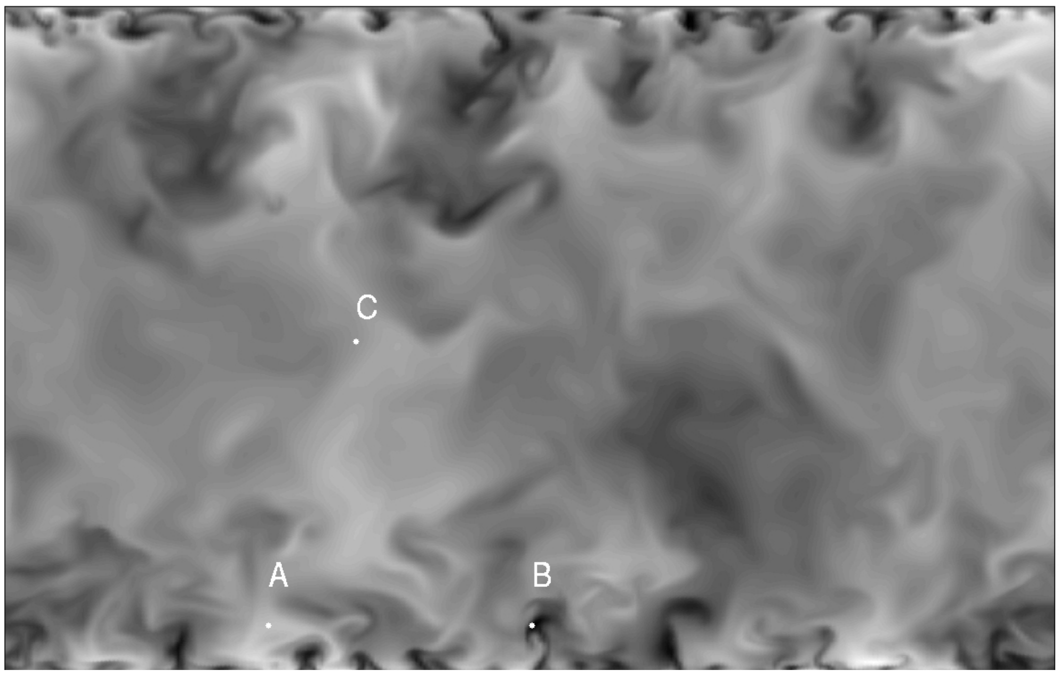


- simulation: 256 Fourier $\times$ 300 B-splines
- spectral interpolation upon L-G-L grid
- analysis with 256 periodic spline wavelets $\times$ 257 Legendre wavelets

Slice from plane channel flow

$$Re_\tau = \frac{h\,u_\tau}{\nu} = 590$$

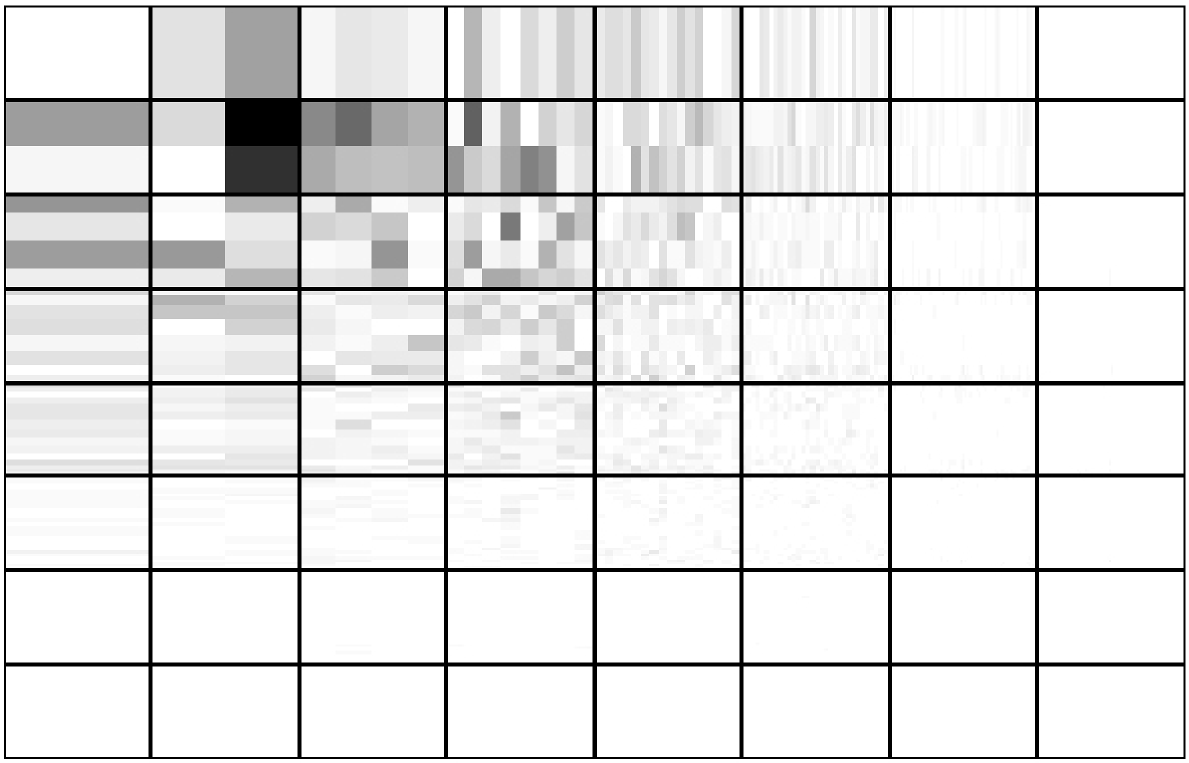
streamwise velocity fluctuations



$\otimes$
(mean flow)

- simulation: 600 Fourier $\times$ 385 Chebyshev
- spectral interpolation upon L-G-L grid
- analysis with 512 periodic spline wavelets $\times$ 513 Legendre wavelets

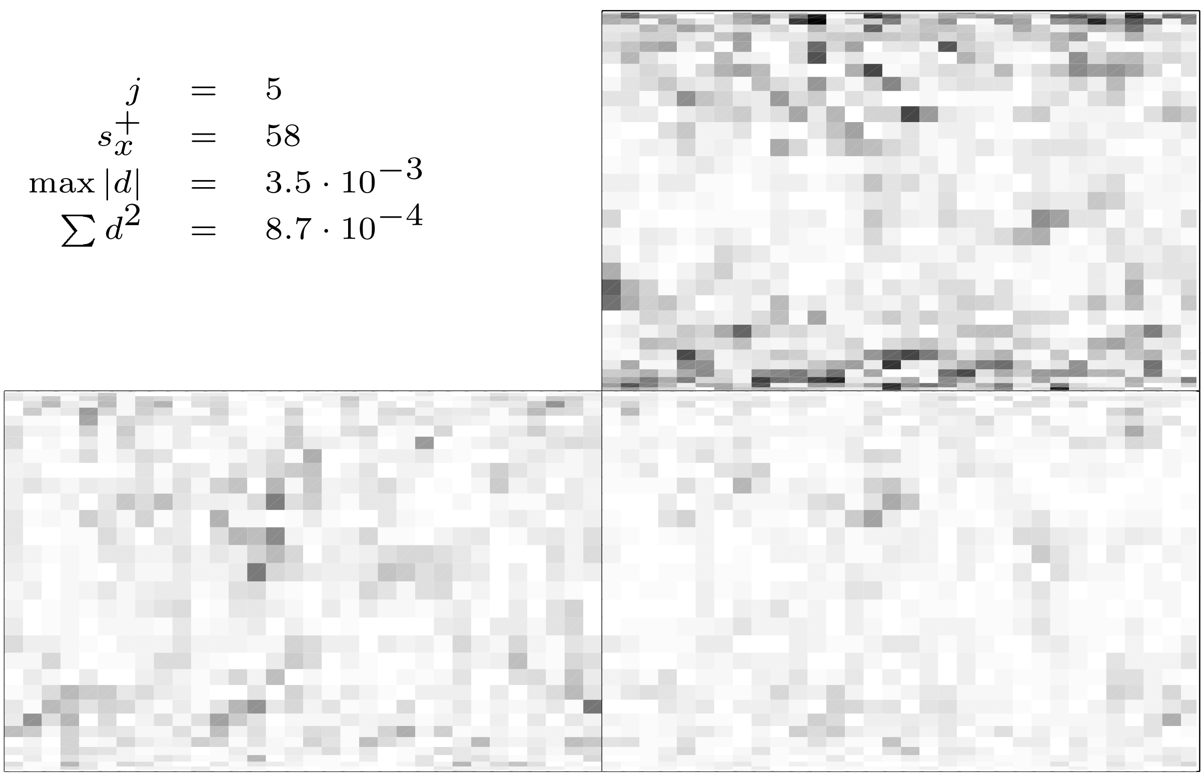
Channel: plain coefficient scheme



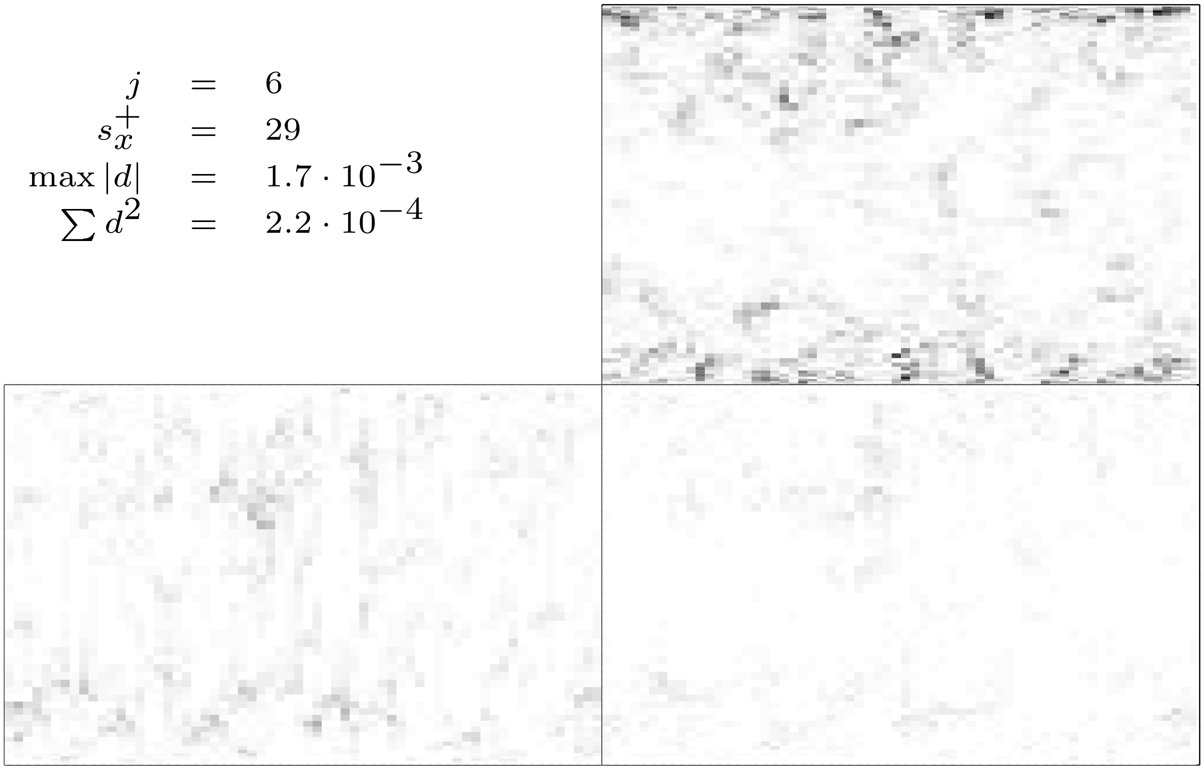
- interpretation not obvious
- need derived quantities

Slice from plane channel flow

$$\begin{aligned} j &= 5 \\ s_x^+ &= 58 \\ \max|d| &= 3.5 \cdot 10^{-3} \\ \sum d^2 &= 8.7 \cdot 10^{-4} \end{aligned}$$

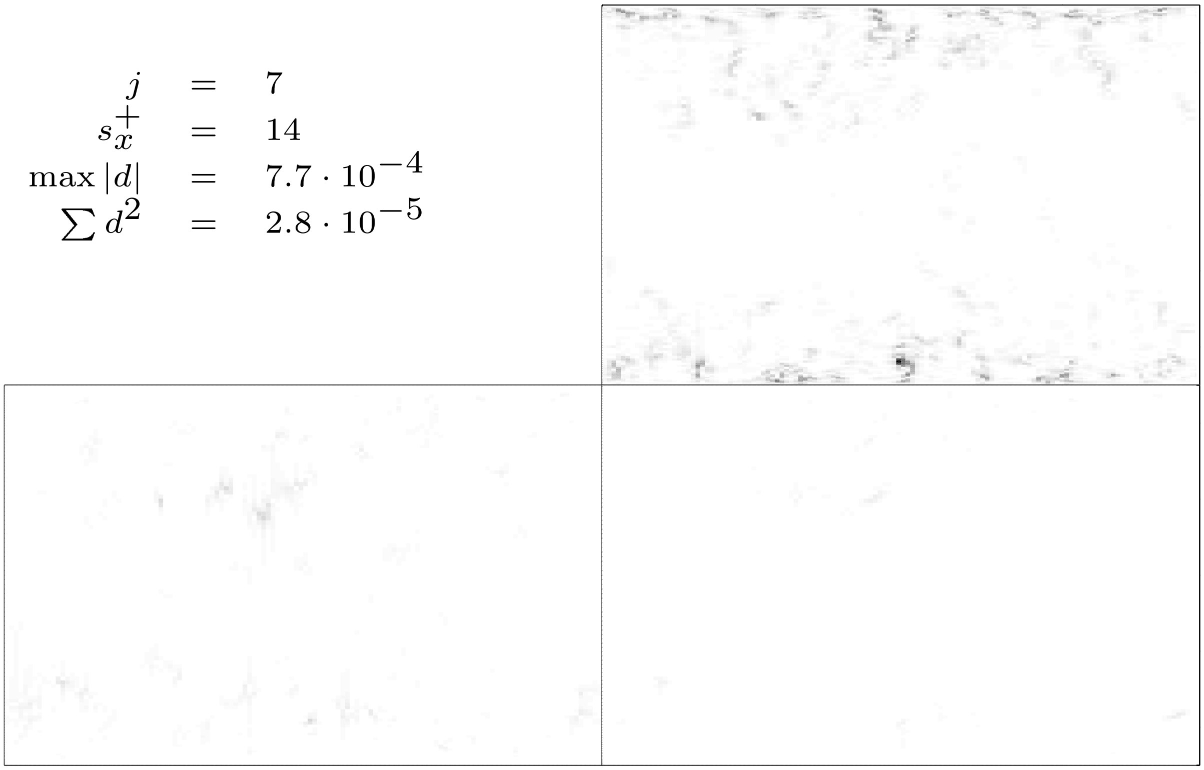


$$
\begin{aligned}
j &= 6 \\
s_x^+ &= 29 \\
\max|d| &= 1.7 \cdot 10^{-3} \\
\sum d^2 &= 2.2 \cdot 10^{-4}
\end{aligned}
$$

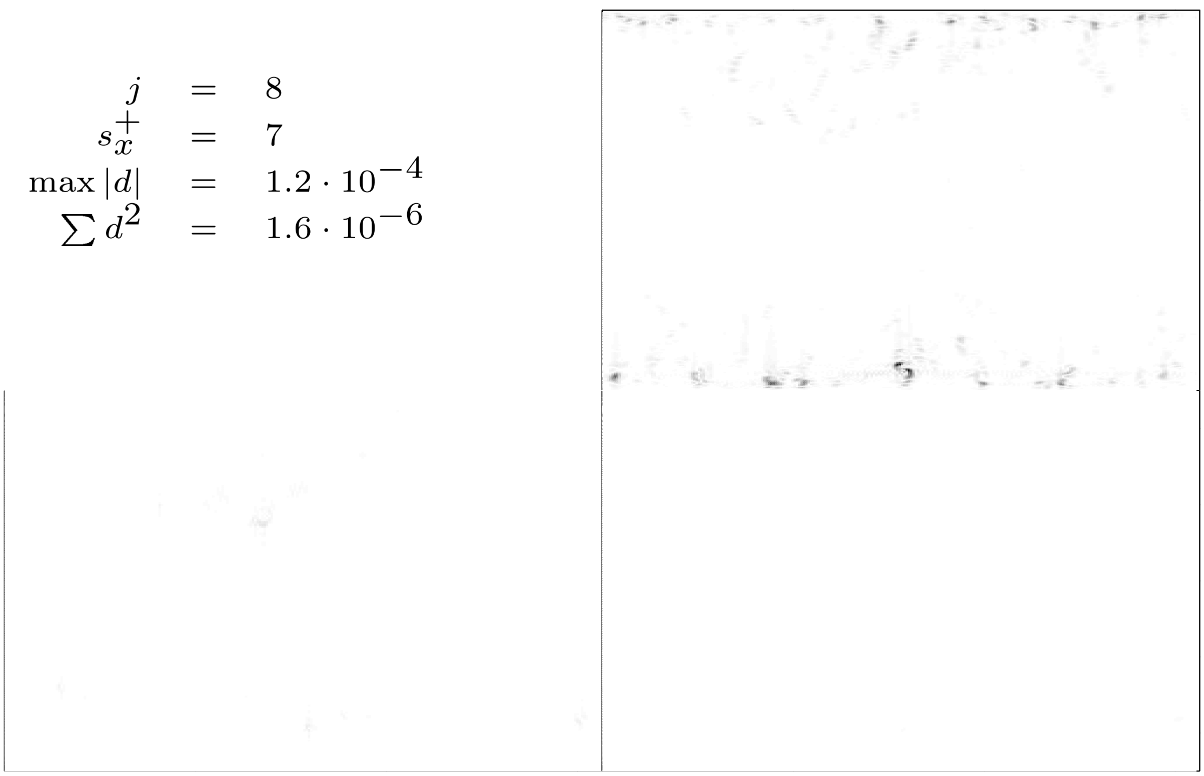


Slice from plane channel flow

$$
\begin{aligned}
j &= 7 \\
s_x^+ &= 14 \\
\max |d| &= 7.7 \cdot 10^{-4} \\
\sum d^2 &= 2.8 \cdot 10^{-5}
\end{aligned}
$$



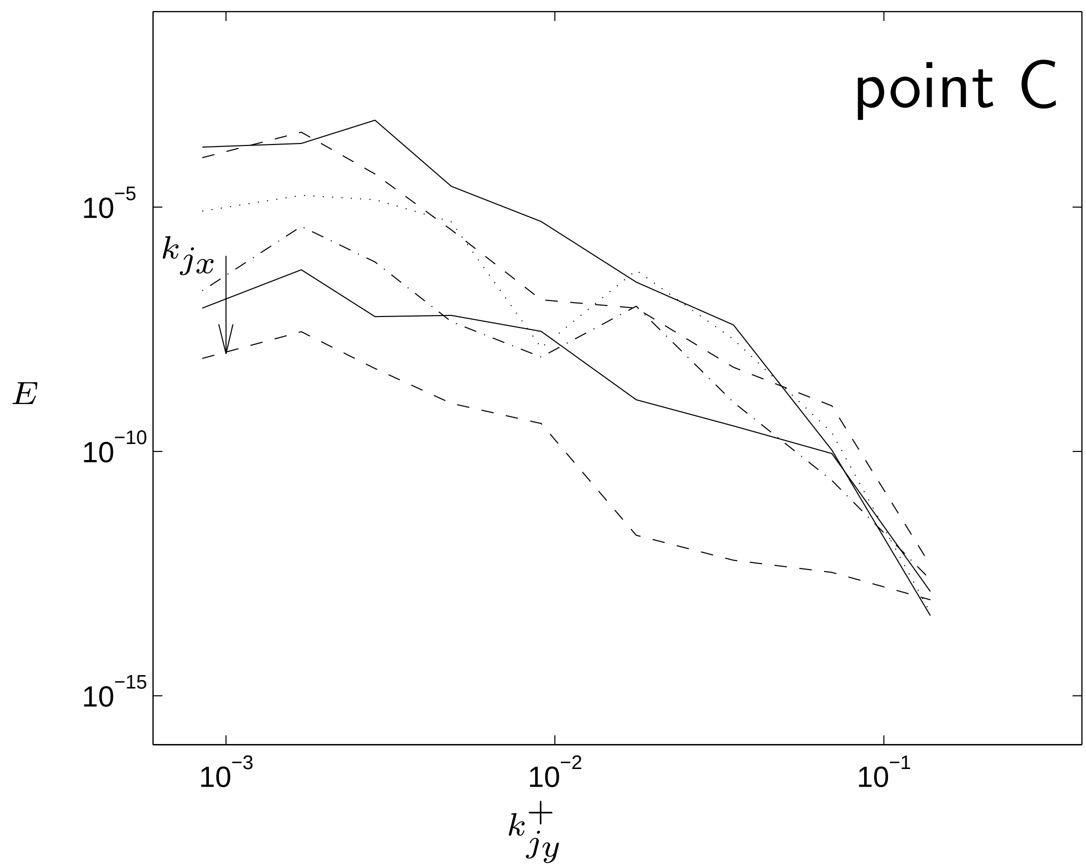
$$
\begin{aligned}
j &= 8 \\
s_x^+ &= 7 \\
\max |d| &= 1.2 \cdot 10^{-4} \\
\sum d^2 &= 1.6 \cdot 10^{-6}
\end{aligned}
$$



Local energy spectra

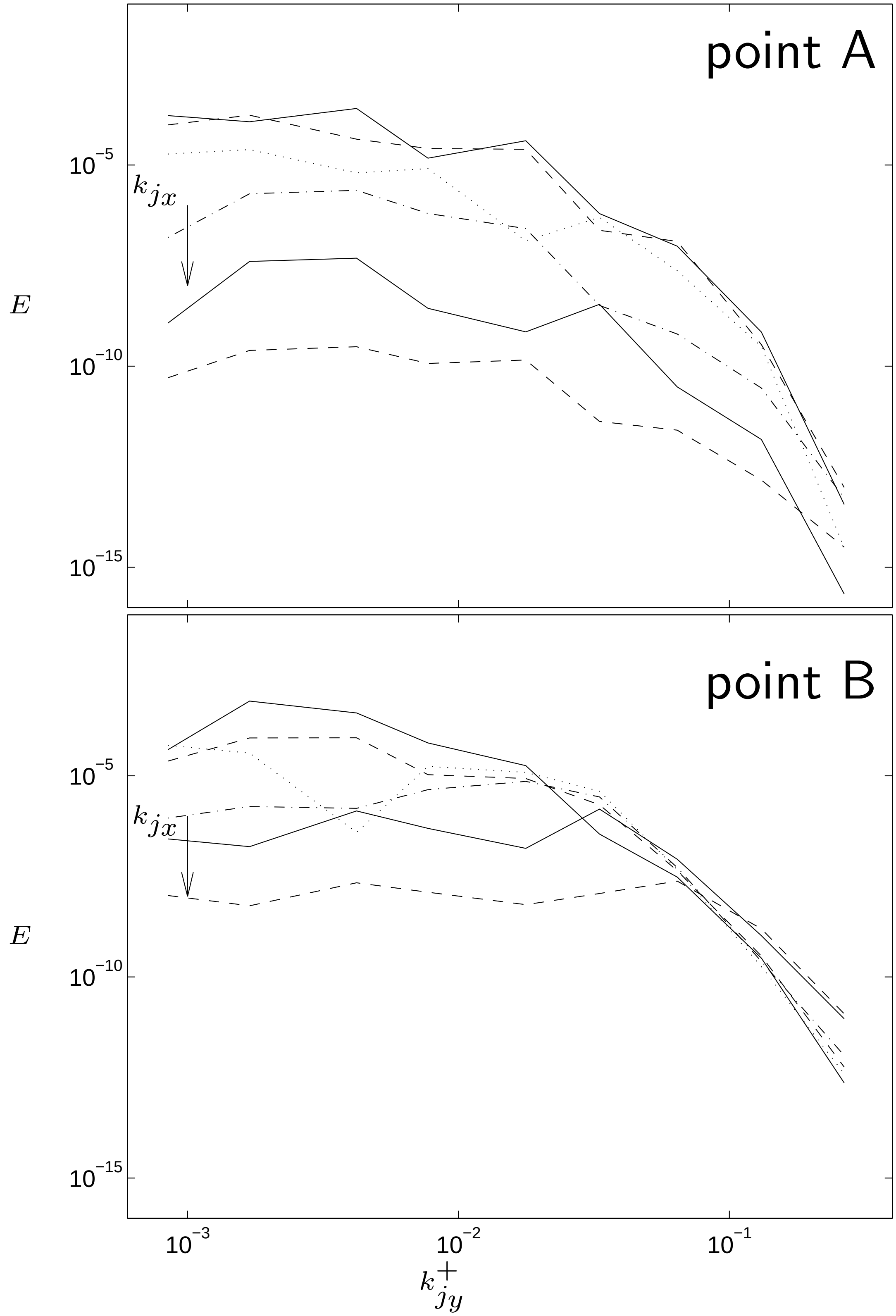
- define local 2d power spectral density

$$E(k_{j_x}, k_{j_y}, x, y) \equiv \frac{\left(d^{j_x, j_y}_{i_{xc}, i_{yc}}\right)^2}{\Delta k_{j_x} \Delta k_{j_y}} 2^{j_x + j_y}$$



$$k^+_{jx} = \{4.3 \cdot 10^{-3}, 8.6 \cdot 10^{-3}, 1.7 \cdot 10^{-2}, 3.5 \cdot 10^{-2}, 6.9 \cdot 10^{-2}, 0.14\}$$

Local energy spectra

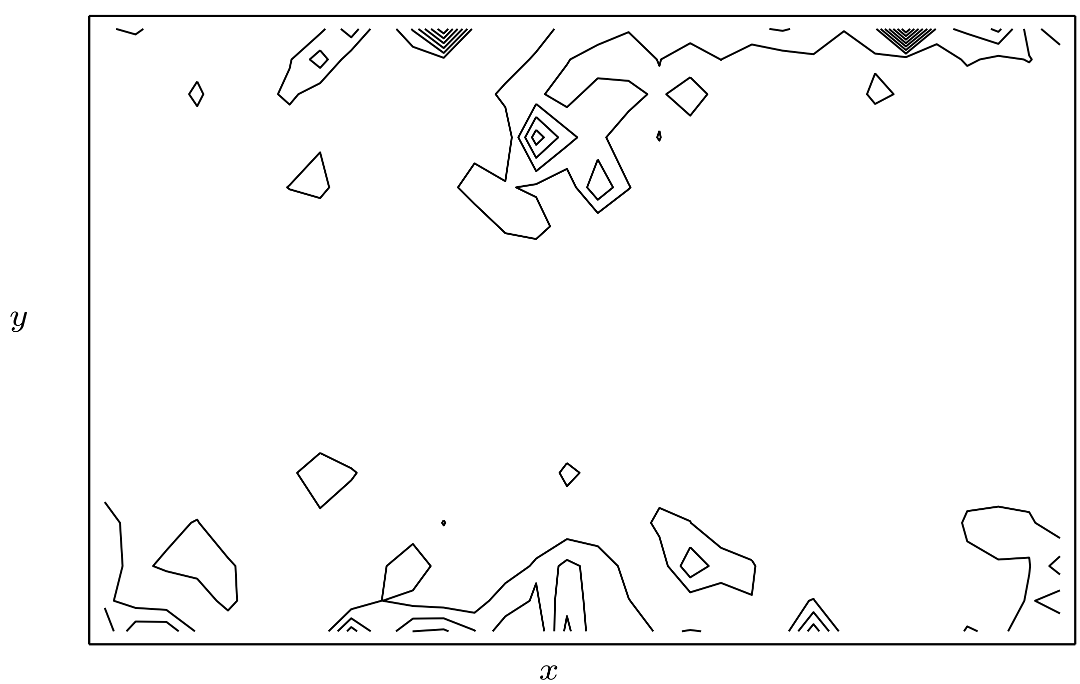


Intermittency index

- define a measure of scale-wise relative energy

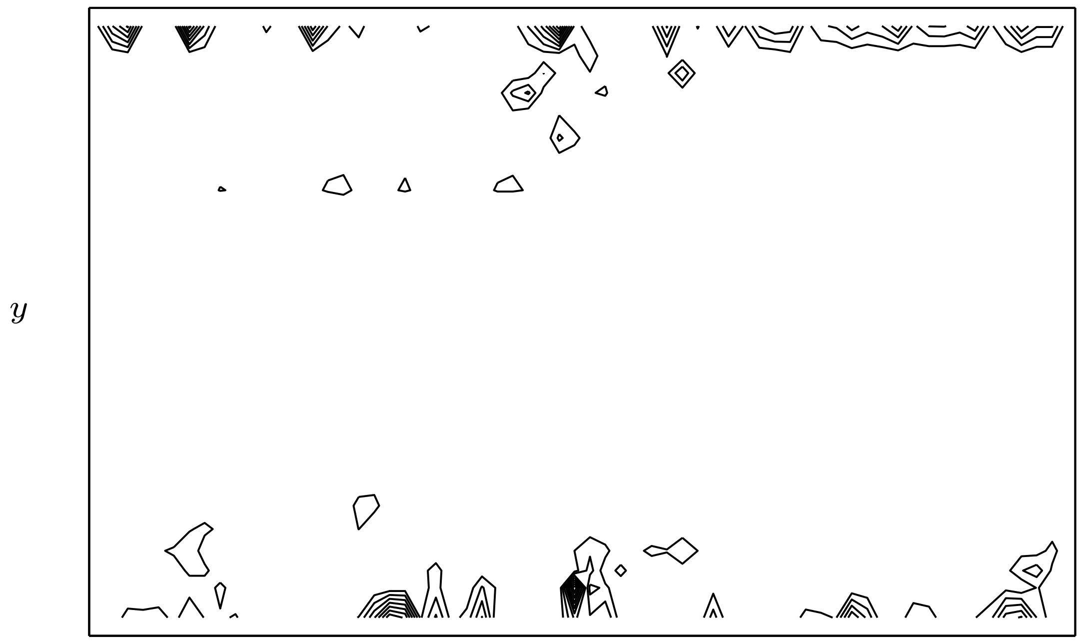
$$I(s_x, s_y, x, y) \equiv \frac{\sum_{\alpha=1}^{3} \left(d_{i_{xc}, i_{yc}}^{j_x, j_y}(u_\alpha) \right)^2}{\sum_{i_{xc}, i_{yc}} \sum_{\alpha=1}^{3} \left(d_{i_{xc}, i_{yc}}^{j_x, j_y}(u_\alpha) \right)^2 / n_y n_x}$$

$s_x^+ = 58,\ s_y^+ \approx 75,\ max(I) = 19$

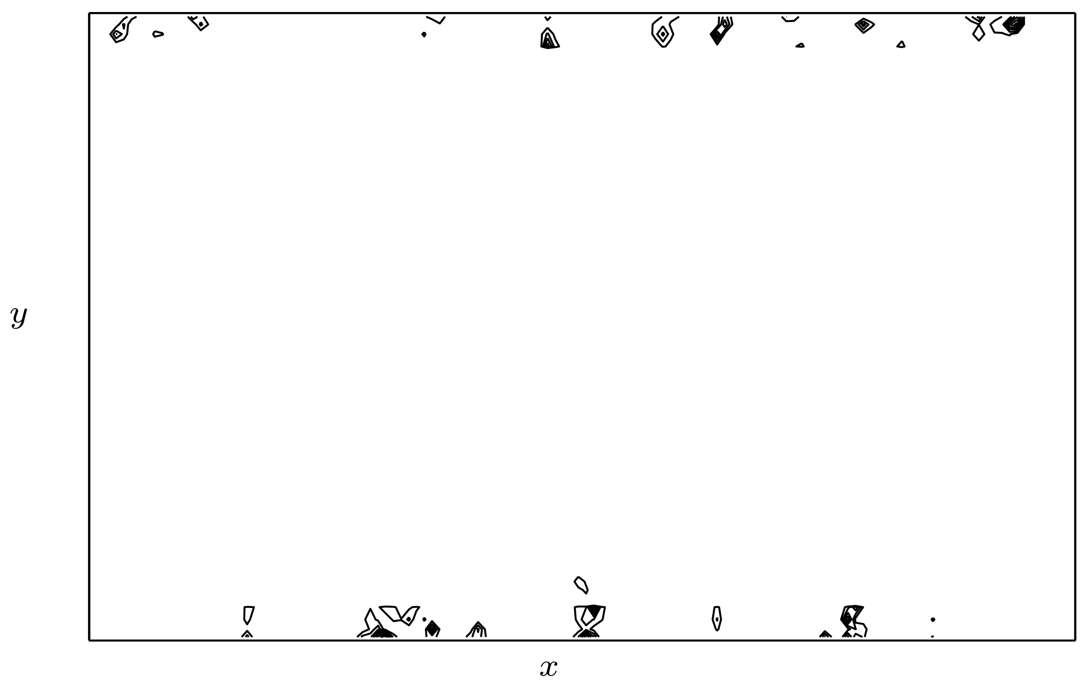


Intermittency index

$$s_x^+ = 29,\ s_y^+ \approx 39,\ max(I) = 44$$



$$s_x^+ = 15,\ s_y^+ \approx 19,\ max(I) = 191$$



Statistics: characteristic wall-normal length scales

- define time-averaged 2d power spectral density, fct. of wall-distance

$$< E_{\alpha\beta}(j_x, j_y, y) > \equiv 2^{j_x} \sum_{i_x=0}^{2^{j_x}-1} \frac{< d_{i_x,i_y}^{j_x,j_y}(u_\alpha) d_{i_x,i_y}^{j_x,j_y}(u_\beta) >}{\Delta k_y \Delta k_x}$$

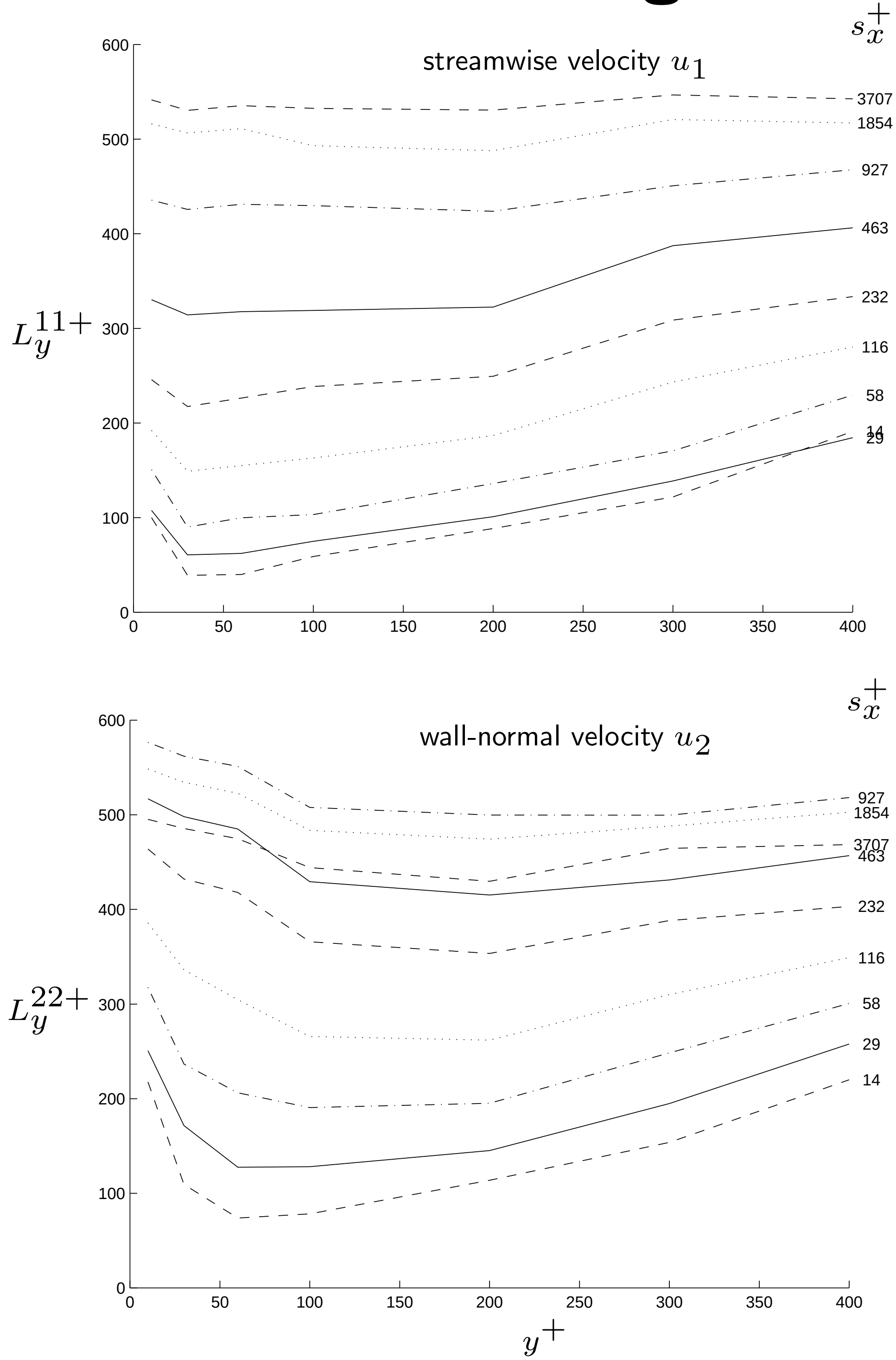
- define char. length of “energetic” scales

$$L_y^{\alpha\beta}(j_x, y) \equiv \frac{\displaystyle\int_0^\infty < E_{\alpha\beta}(j_x, j_y, y) > k_y^{-1} \, \mathrm{d}k_y}{\displaystyle\int_0^\infty < E_{\alpha\beta}(j_x, j_y, y) > \, \mathrm{d}k_y}$$

- define char. length of “dissipative” scales

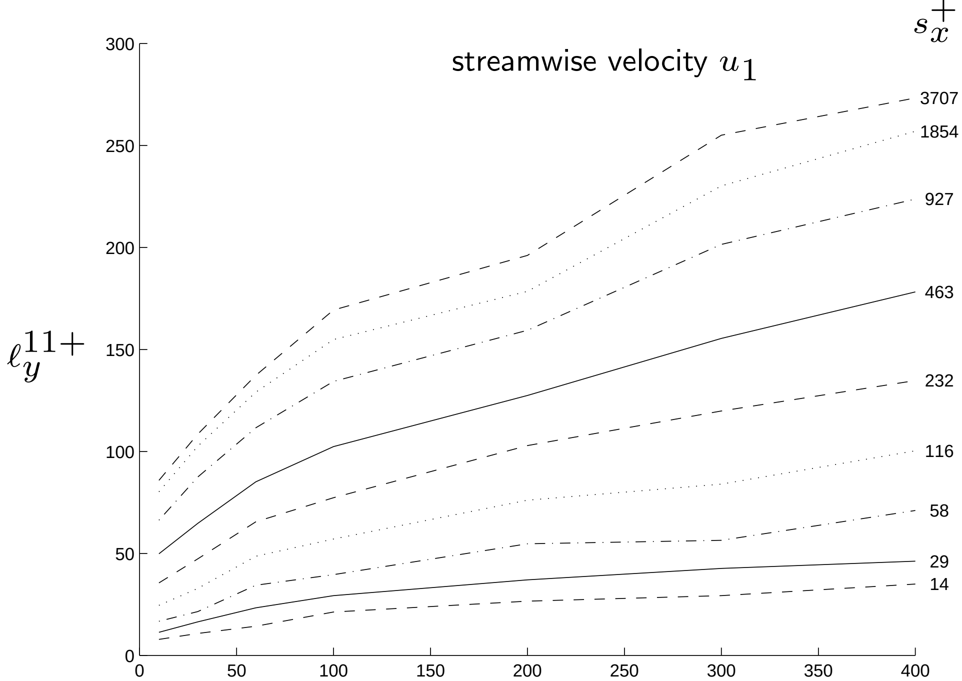
$$\left(\ell_y^{\alpha\beta}(j_x, y)\right)^{-2} \equiv \frac{\displaystyle\int_0^\infty < E_{\alpha\beta}(j_x, j_y, y) > k_y^2 \, \mathrm{d}k_y}{\displaystyle\int_0^\infty < E_{\alpha\beta}(j_x, j_y, y) > \, \mathrm{d}k_y}$$

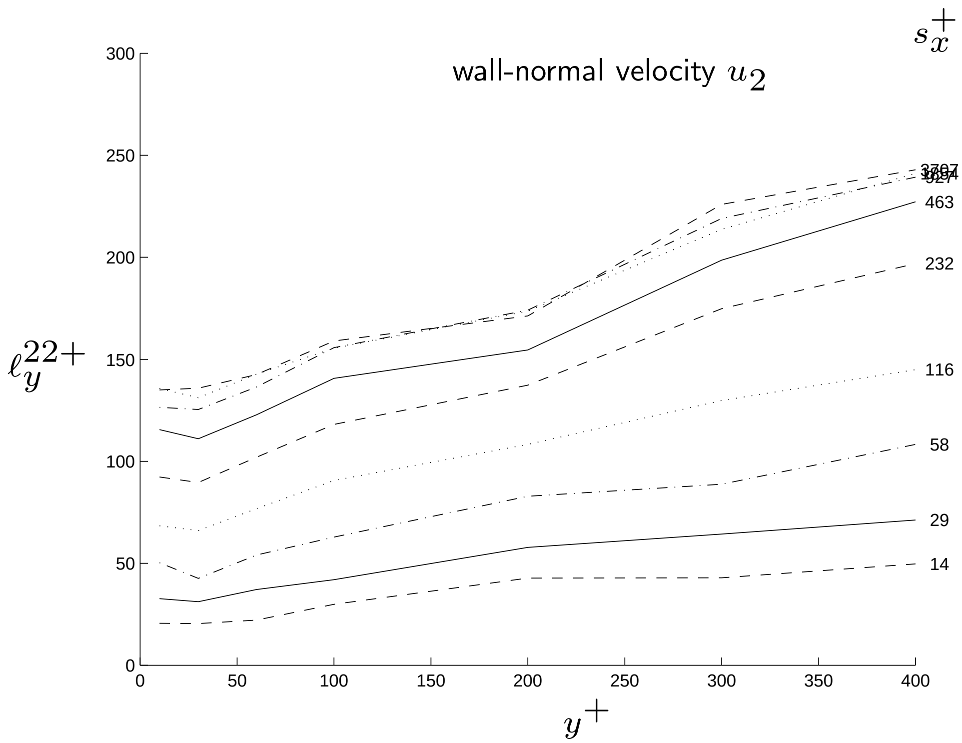
Wall-normal energetic scales



- mild y-dependence; L_y^{11}, L_y^{22} similar

Wall-normal dissipative scales





- stronger y-dependence; ℓ_y^{11}, ℓ_y^{22} similar

Recap: multi-d bases

- extension of discrete transform by tensor product of 1d functions
- analysis/visualization more intricate
- method (C): qualitative; directional index and aspect ratio interfere
- method (B): preferred; direction through sole aspect ratio
- diagnostic tools: 2d local spectra, intermittency index, integral scales, ...

Transform of vector quantities

standard: use scalar wavelets for each component $f_i(\mathbf{x})$ of vector field $\mathbf{f}(\mathbf{x})$

divergence-free data: (e.g. incompressible flow)

- filtered field *does not* remain div-free
- analysis of *mode-wise* non-linear transfer processes (e.g. energy, enstrophy) not possible

$\Rightarrow$ need genuinely div-free wavelets

Helical wavelets, Kishida (2000)

(i) helical decomposition of Fourier comp.

$$\hat{\mathbf{f}}(\mathbf{k}) = \hat{f}_+ \mathbf{h}_+(\mathbf{k}) + \hat{f}_- \mathbf{h}_-(\mathbf{k}) \left(+ \hat{f}_0 \mathbf{h}_0(\mathbf{k}) \right)$$

(ii) "helical pull-up" $\hat{\boldsymbol{\psi}}_\pm(\mathbf{k}) = \hat{\psi} \mathbf{h}_\pm(\mathbf{k})$

$\Rightarrow$ union of pull-ups of wavelet basis

$$\mathbf{f}(\mathbf{x}) = \sum_{\lambda \in \Lambda} f_\lambda \boldsymbol{\psi}_\lambda(\mathbf{x}), \; f_\lambda = \int \mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\psi}_\lambda(\mathbf{x}) \mathrm{d}\mathbf{x}$$

- indices $\Lambda = \{j;\, i;\, 1 \le q \le 7;\, s \in \{+,-\}\}$
- div-free (mode-wise): $\nabla \cdot \boldsymbol{\psi}_\lambda(\mathbf{x}) = 0$

simple algorithm:

(i)	FFT forward	$\mathbf{f}(\mathbf{x}) \to \hat{\mathbf{f}}(\mathbf{k})$
(ii)	scalar product	$\mathbf{h}_s^*(\mathbf{k}) \cdot \hat{\mathbf{f}}(\mathbf{k})$
(iii)	FFT backwards	$\mathbf{h}_s^*(\mathbf{k}) \cdot \hat{\mathbf{f}}(\mathbf{k}) \to f_s(\mathbf{x})$
(iv)	fast (scalar) WT	$f_\lambda = < f_s, \psi_{ji}^q >$

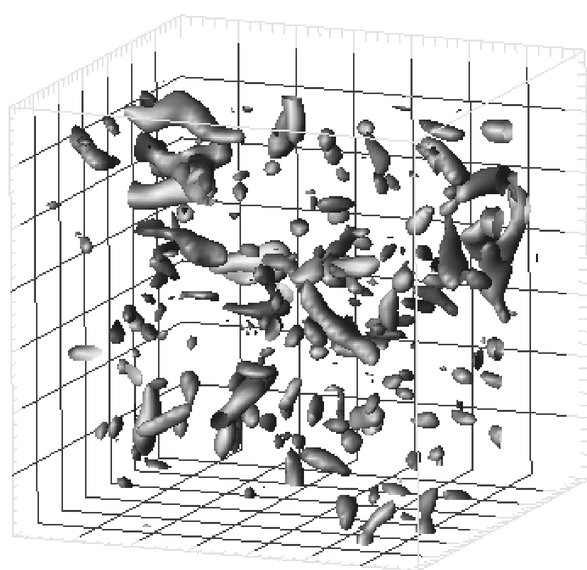
Helical wavelets & filtering

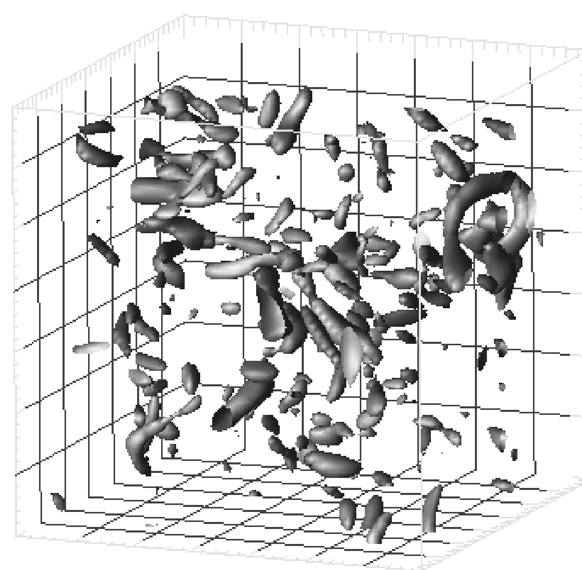
- triply-periodic physical space

here: used together with spline wavelets

- localization similar
- compression properties similar
- scale filtering:

scalar wavelets (divergence!) helical wavelets





enstrophy of freely-decaying, homogeneous, isotropic turbulence, $Re_\lambda = 20$, $N = 64^3$. scale of filter: $s_j = 0.7\lambda$ (band-pass).

Helical wavelets & non-linear transfer

evolution of energy of individual mode "a"

$$\mathrm{d}_t e^a - \nu u^a \sum_d u^d \int \boldsymbol{\psi}^a \cdot \nabla^2 \boldsymbol{\psi}^d \mathrm{d}\mathbf{x} =$$

$$- \quad u^a \sum_b \sum_c u^b u^c \underbrace{\int \boldsymbol{\psi}^a \cdot \left(\boldsymbol{\psi}^b \cdot \nabla\right) \boldsymbol{\psi}^c \mathrm{d}\mathbf{x}}_{M_{abc}}$$

- "balancing pair" (analog Fourier triad)

$$M_{abc} + M_{cba} = 0$$

transfer of energy between $a \leftrightarrow c$, but depending on b

in practice: size prohibitive for evaluation

$\longrightarrow$ need *a priori* reduction

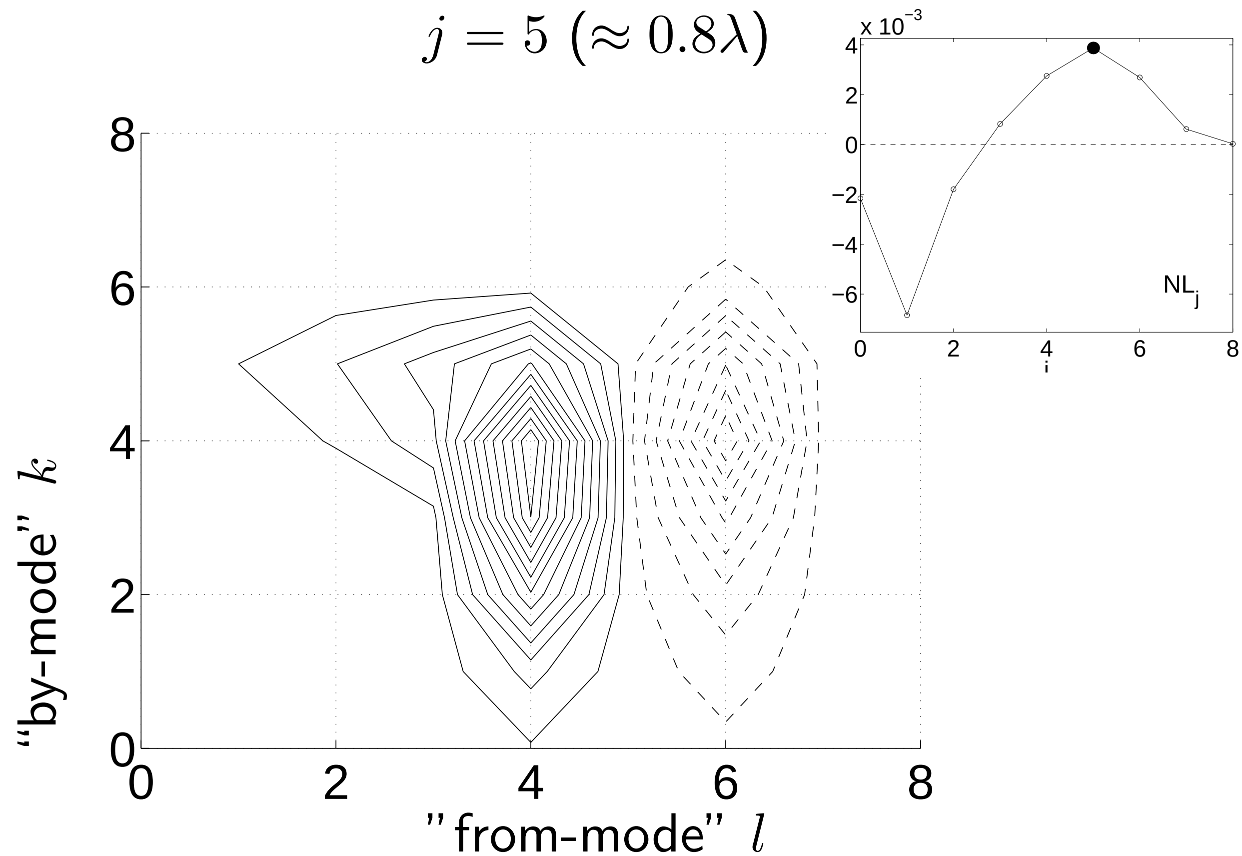
Inter-scale energy transfer

<u>reduce over</u>: position, direction, helicity

$$\mathbf{u}^{(j)}(\mathbf{x}) = \sum_{q,i,s} u^\lambda \, \boldsymbol{\psi}^\lambda(\mathbf{x})$$

non-linear term:

$$NL_j = -\sum_k \sum_l \int \mathbf{u}^{(j)} \cdot \left(\mathbf{u}^{(k)} \cdot \nabla\right) \mathbf{u}^{(l)} \mathrm{d}\mathbf{x}$$



forced, homogeneous, isotropic turbulence, $Re_\lambda = 150$, $N = 512^3$.

Work in progress – extensions

- analyze enstrophy budget:
  separate advection from vortex stretching

- use different reduction schemes:
  e.g. localized in space, conditioned upon certain “events”

Parallel version of Mallat's algorithm

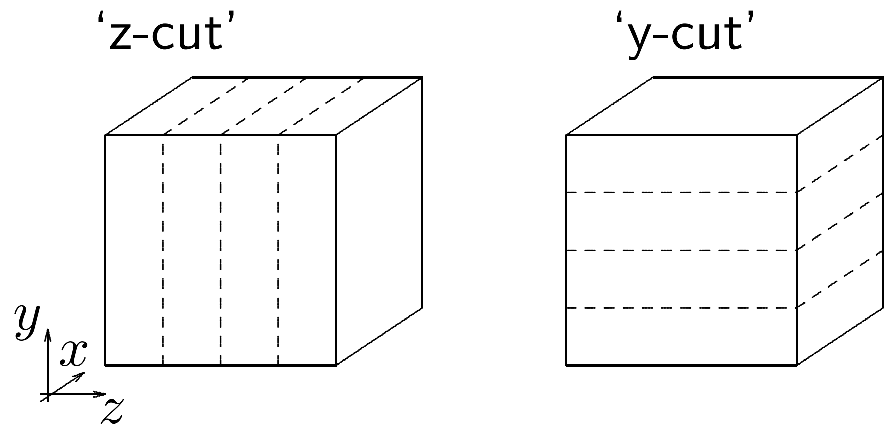
- huge datasets from 3D simulations (512^3 modes)
- multi-processor machines (suppose 2^m procs)
- "long" filters: work in Fourier space

ingredients:
- FFT
- convolution
- down-/upsample

additionally:
- transpose of data
- vary number of active processors

Distribution of data/work

(i) use classical "slice" data model



(ii) at each step of Mallat algorithm,
data-size is halved

$\longrightarrow$ need to de-activate processors

	0	1	2	3	4	5	6	7	i_p
$\log_2(N)-1$	●	●	●	●	●	●	●	●	
			$\vdots$						
3	●	●	●	●	●	●	●	●	
2	●	○	●	○	●	○	●	○	
1	●	○	○	○	●	○	○	○	
0	●	○	○	○	○	○	○	○	
scale j									

Performance

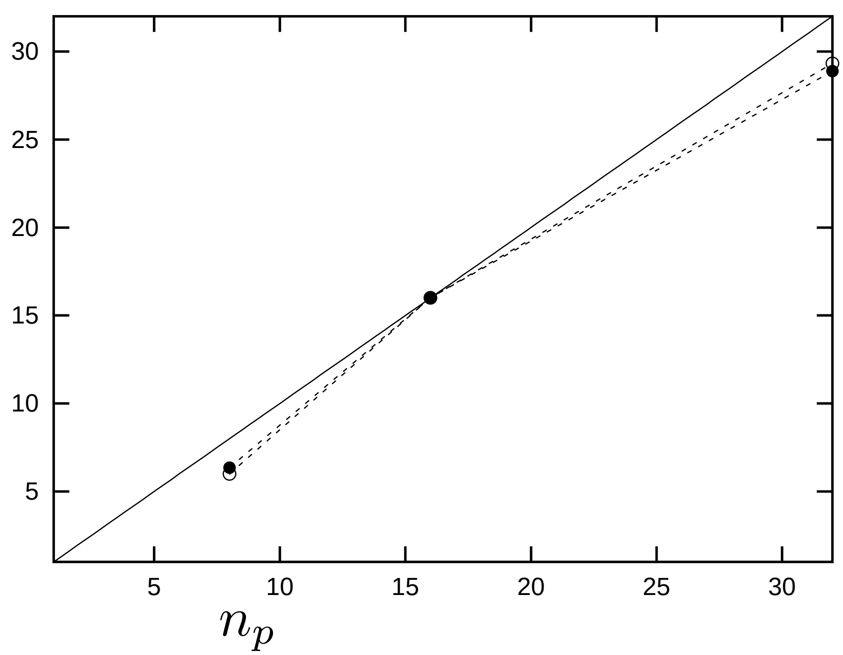
(i) memory

- nearly inv. proportional to no. procs n_p
- bias scales as $\Delta n = (n_p^2 - 1) \cdot 7/3$
- retrieval scheme for individual coefficients

(ii) execution

- solid speed-up (e.g. $N = 256^3$)

$S \equiv \frac{\Delta t(1)}{\Delta t(n_p)}$



sum of forward and inverse transform;

IBM SP2 & CRAY T3E

General conclusion

progress in various aspects:

- polynomial wavelets for the interval
- analysis using divergence-free functions
- feasible computation for large data sets

however, still not ready to meet all requirements in the most general case

analysis criteria:

- ○ orthogonality
- ○ localization
- ○ smoothness
- ○ symmetry
- ○ fast algorithm

data from turbulence:

- ○ divergence-free
- ○ 3-dimensional
- ○ non-homogeneous (bounded)