1 Vortex dipole rebound from a wall July 3, 2001

1.1 The initial field

We consider a vortex dipole according to the inviscid solution of Batchelor [1, p. 424ff., p. 535ff.]. The streamfunction reads in polar coordinates:

$$\psi(r,\theta) = -C \cdot J_1(kr) \cdot \cos(\theta)\,, \qquad r \le a \quad , \tag{1}$$

where C is an arbitrary real constant fixing the amplitude, k is a parameter which will determine the characterisitic radius of the vortex pair, and $r = \sqrt{(x-x_0)^2 + (x-y_0)^2}$ is the radius relative to a center (x_0, y_0); J_1 refers to the Bessel function of the first kind.

This solution is matched to an irrotational outer solution for $r > a$ at the radius where the vorticity vanishes, i.e. where $J_1(ka) = 0$, leading to:

$$a = k \cdot 3.8317\ldots \quad . \tag{2}$$

In our simulation code we need to supply initial values for the Laplacian of the velocity in the y-coordinate, say v. Since basic vector identities [e.g. 2, p. 57] imply that $\nabla^2\vec{v} = \nabla(\nabla.\vec{v}) - \nabla \times (\vec{\omega})$ and the present velocity field $\vec{v}$ is incompressible and irrotational for $r > a$, we have

$$\varphi \equiv \nabla^2 v = 0\,, \qquad r > a \quad . \tag{3}$$

Otherwise ($r \le a$), we first calculate the two planar velocity components w, v, viz:

$$w = \sin(\theta) C \left(J_0(kr) - \frac{J_1(kr)}{kr} \right) k \cos(\theta) - \frac{\sin^2(\theta) C J_1(kr)}{r}\,, \tag{4a}$$

$$v = -\cos^2(\theta) C \left(J_0(kr) - \frac{J_1(kr)}{kr} \right) k - \frac{\sin^2(\theta) C J_1(kr)}{r}\,, \tag{4b}$$

and then the Laplacian of v by taking into account the variable transformation $x = |\vec{r}|\cos(\theta)$, $y = |\vec{r}|\sin(\theta)$ [e.g. 3, p. 285]:

$$\varphi(x,y) = \frac{Ck^2}{r^3}\left(krx^2 J_0(kr) - x^2 J_1(kr) - y^2 J_1(kr)\right)\,, \quad r \le a \quad . \tag{5}$$

Incidentally, the vorticity component perpendicular to the plane, say ω_x, has the following functional form:

$$\omega_x = \frac{1}{r} C k^2 x J_1(kr)\,, \qquad r \le a \quad . \tag{6}$$

The initial velocity of displacement of the vortex pair is given by:

$$v_\infty = -\frac{1}{2} C k J_0(ka) \approx -0.2014\ldots Ck \quad . \tag{7}$$

1.2 Parameters used in our simulation

As in [4] we define a Reynolds number through the following characteristics of the dipole:

$$Re = \frac{|v_\infty| a}{\nu} = \frac{CkaJ_0(ka)}{2\nu} \approx \frac{C}{\nu}\, 0.77163\ldots \quad . \tag{8}$$

The constants are adjusted such that we have $Re = 771$ in the following. A characteristic time-scale can be defined as follows:

$$t_{ref} = a|v_\infty| = \frac{2a}{CkJ_0(ka)} \approx \frac{1}{C}\, 19.03\ldots \quad . \tag{9}$$

1.3 Numerical set-up

We carried out the simulation of the motion of the dipole by using a three-dimensional DNS code for plane channel flow. The plane of the vortex motion is a spanwise section of the channel, with all quantities set constant in the streamwise direction. $N_z = 256$ Fourier modes were used to discretize the z-direction and $N_y = 300$ B-splines in the y-direction (using a tangent hyperbolic stretching by a factor of approximately 11). Voriticity isolines at several instants during the complex evolution can be seen in figure 2.

References

[1] G.K. Batchelor. *An introduction to fluid dynamics.* Cambridge University Press, 1967.

[2] R. Aris. *Vectors, Tensors, and the Basic Equations of Fluid Mechanics.* Dover Science and Maths, 1962.

[3] I.N. Bronstein and K.A. Semendjajew. *Taschenbuch der Mathematik*, volume 1. Harri Deutsch, 24 edition, 1989.

[4] P. Orlandi. Vortex dipole rebound from wall. *Phys. Fluids A*, 2(8):1429–1436, 1990.

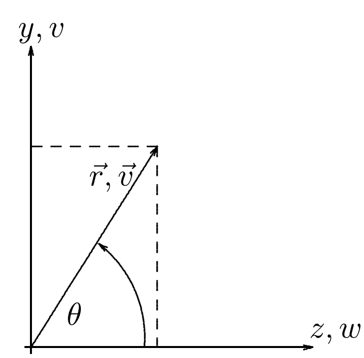

Figure 1: Polar and cartesian coordinate system definitions.

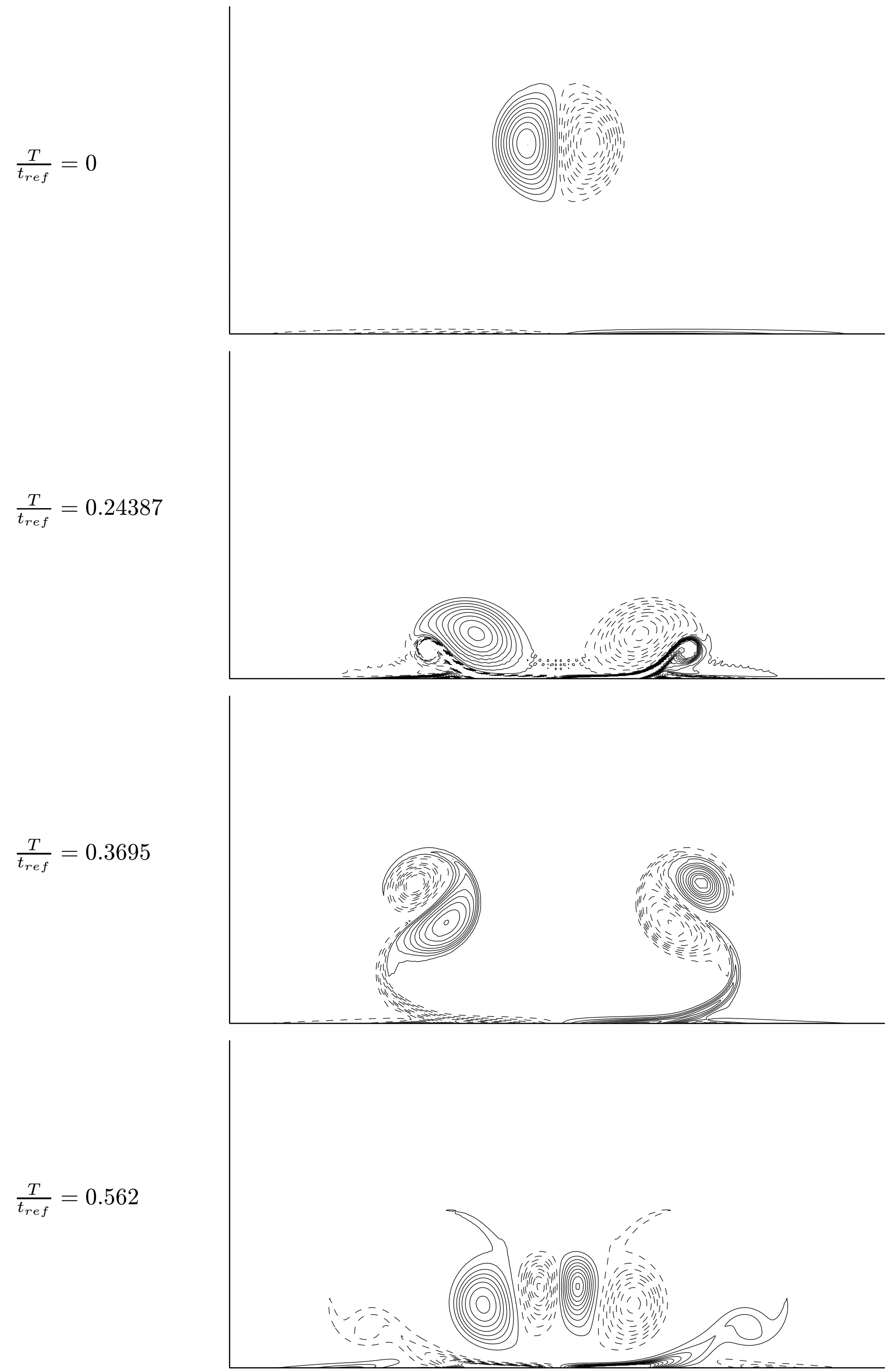

Figure 2: Isocontourlines of vorticity at 20 linearly-spaced values in the range between the maximum and minimum of the initial field, negative values having a dashed line-style.