Turbulenzmodelle in der Strömungsmechanik

Turbulent flows and their modelling

Markus Uhlmann

Institut für Hydromechanik

www.ifh.uni-karlsruhe.de/people/uhlmann

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LECTURE 9

k–ε and other eddy viscosity models

Questions to be answered in the present lecture

How can the turbulent viscosity/Reynolds stress be determined from two field equations?

1. k–ε model
2. k–ω model

Does this improve the predictive capability?

Overview of two-equation models

Features of two-equation models

- Boussinesq hypothesis is used to express the Reynolds stress
- transport equations are solved for *two* turbulent scales (ϕ, ψ)
- turbulent viscosity is constructed from these scales:
$$\nu_T \sim \phi^n \cdot \psi^m$$
- powers m, n from dimensional consistency
- specification of case-dependent length scale not necessary

⇒ should provide greater universality

Variants of two-equation models

k-ε model

- dissipation rate ε used to construct a length scale
- most widespread two-equation model
- extensively used in commercial codes

k-ω model

- ω: characteristic frequency of energy containing eddies
- has some advantages in wall-bounded flows

- other choices for the second transported scale: τ, ℓ, $k \cdot \ell$, ...

Ingredients of the k-ε model

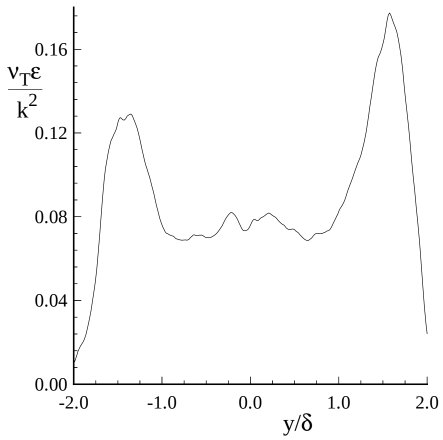
1. the Boussinesq hypothesis: $\langle u_i' u_j' \rangle = -2\nu_T \bar{S}_{ij} + \frac{2}{3} k \, \delta_{ij}$
2. the expression for the turbulent viscosity: $\nu_T = C_\mu k^2 / \varepsilon$
   with a constant $C_\mu = 0.09$
3. the transport equation for k (cf. lecture 8)
4. the transport equation for the dissipation rate ε
5. initial & boundary conditions
6. (unfortunately) additional modifications ...

$\rightarrow$ the main task is to model the ε equation

Turbulent viscosity "constant" C_μ in simple shear flows

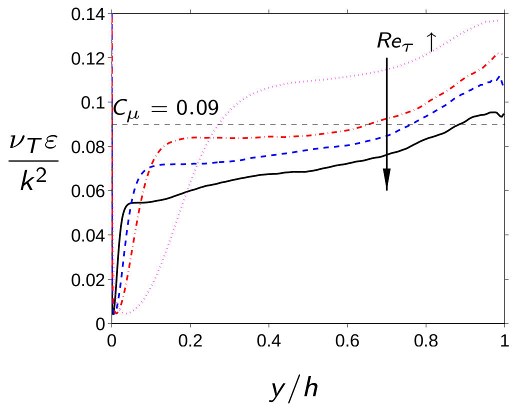
temporal mixing layer

(DNS Rogers & Moser 1994)



channel flow $Re_\tau = 180 \ldots 2000$

(DNS Jimenez et al.)



- C_μ roughly constant (but not near wall or freestream)
- value $C_\mu = 0.09$ a compromise (Reynolds number effects)

Model equation for turbulent kinetic energy

Recall the modelled TKE transport equation (cf. lecture 8)

$$\frac{\bar{\mathrm{D}}k}{\bar{\mathrm{D}}t} - \mathcal{P} = \nabla \cdot \left(\left(\nu + \frac{\nu_T}{\sigma_k} \right) \nabla k \right) - \tilde{\varepsilon}$$

- with: $\sigma_k = 1$
- production term $\mathcal{P} = -\langle u_i' u_j' \rangle \, \langle u_i \rangle_{,j}$
  $\longrightarrow$ closed through turbulent viscosity expression
- open term: pseudo-dissipation rate $\tilde{\varepsilon}$

Model equation for the turbulent dissipation rate

$$\frac{\bar{\mathrm{D}}\tilde{\varepsilon}}{\bar{\mathrm{D}}t} = -2\nu\left(\langle u'_{i,k}u'_{j,k}\rangle + \langle u'_{k,i}u'_{k,j}\rangle\right)\langle u_i\rangle_{,j} - 2\nu\langle u'_k u'_{i,j}\rangle\langle u_i\rangle_{,kj}$$
$$-2\nu\langle u'_{i,k}u'_{i,m}u'_{i,m}\rangle - 2\nu^2\langle u'_{i,km}u'_{i,km}\rangle$$
$$+\left(\nu\tilde{\varepsilon}_{,j} - \nu\langle u'_j u'_{i,m}u'_{k,m}\rangle - 2\frac{\nu}{\rho}\langle u'_{j,m}p'_{,m}\rangle\right)_{,j}$$

Exact equation for the pseudo-dissipation rate

- an exact equation can be derived from Navier-Stokes

$\rightsquigarrow$ modeling term-by-term is considered unfeasible

$\Rightarrow$ use an entirely modeled equation instead!

Model equation for the turbulent dissipation rate (2)

Artificial equation for the pseudo-dissipation rate

$$\frac{\bar{\mathrm{D}}\tilde{\varepsilon}}{\bar{\mathrm{D}}t} = \nabla\cdot\left(\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\nabla\tilde{\varepsilon}\right) + \mathcal{P}\frac{C_{\varepsilon_1}\tilde{\varepsilon}}{k} - \tilde{\varepsilon}\frac{C_{\varepsilon_2}\tilde{\varepsilon}}{k}$$

- completely analogous to the modeled TKE equation: convection, diffusion, production & destruction effects
- with coefficients: $\sigma_\varepsilon = 1.3$, $C_{\varepsilon_1} = 1.44$, $C_{\varepsilon_2} = 1.92$
- the behavior of the $\tilde{\varepsilon}$ equation is discussed in the following

Model predictions in idealized flows

Model equations in homogeneous flow

$$\boxed{\frac{\mathrm{d}k}{\mathrm{d}t} = \mathcal{P} - \tilde{\varepsilon}\,, \qquad \frac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}t} = \mathcal{P}\frac{C_{\varepsilon 1}\tilde{\varepsilon}}{k} - \frac{C_{\varepsilon 2}\tilde{\varepsilon}^2}{k}}$$

The case of decaying turbulence

- no mean strain $\quad\rightarrow\quad \mathcal{P} = 0$

  $\Rightarrow \qquad \dfrac{\mathrm{d}k}{\mathrm{d}t} + \tilde{\varepsilon} = 0\,, \qquad \dfrac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}t} + \dfrac{C_{\varepsilon_2}\tilde{\varepsilon}^2}{k} = 0$
- solution: $k(t) = k_0(t/t_0)^{-n}, \quad \tilde{\varepsilon}(t) = \tilde{\varepsilon}_0(t/t_0)^{-(n+1)}$

  with: $n = 1/(C_{\varepsilon_2} - 1)$
- experiments show: $n \approx 1.3 \Rightarrow C_{\varepsilon_2} = 1.77$
- slightly lower than standard k-ϵ model value ($C_{\varepsilon_2} = 1.92$)

k-ε model predictions in homogeneous flow (2)

Homogeneous shear flow turbulence

- single mean strain component: $\langle u_i \rangle_{,j} = S\delta_{i1}\delta_{j2}$ with $S = cst$

$\rightarrow \mathcal{P} = -\langle u'v' \rangle\, S$

$\Rightarrow \dfrac{\mathrm{d}k}{\mathrm{d}t} = -\langle u'v' \rangle\, S - \tilde{\varepsilon}\,, \qquad \dfrac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}t} = -\langle u'v' \rangle\, C_{\varepsilon_1}\dfrac{S\tilde{\varepsilon}}{k} - C_{\varepsilon_2}\dfrac{\tilde{\varepsilon}^2}{k}$

- $\dfrac{\mathrm{d}(k/\tilde{\varepsilon})}{\mathrm{d}t} = -\langle u'v' \rangle\, \dfrac{S}{\tilde{\varepsilon}}(1 - C_{\varepsilon_1}) + C_{\varepsilon_2} - 1$
- asymptotic state: $\dfrac{\mathrm{d}(k/\tilde{\varepsilon})}{\mathrm{d}t} = 0 \Rightarrow \langle u'v' \rangle\, \frac{S}{\tilde{\varepsilon}} = \frac{C_{\varepsilon_2}-1}{1-C_{\varepsilon_1}} = -2.09$
- experiments show higher values: $\langle u'v' \rangle\, \frac{S}{\tilde{\varepsilon}} = -1.8\ldots -1.7$

Dissipation equation in the logarithmic region

Consider fully developed channel flow (high Reynolds number)

$$0=\frac{\mathrm{d}}{\mathrm{d}y}\left(\left(\not{\nu}+\frac{\nu_T}{\sigma_k}\right)\frac{\mathrm{d}k}{\mathrm{d}y}\right)+\mathcal{P}-\tilde{\varepsilon}\,,\quad 0=\frac{\mathrm{d}}{\mathrm{d}y}\left(\left(\not{\nu}+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}y}\right)+\mathcal{P}\frac{C_{\varepsilon_1}\tilde{\varepsilon}}{k}-\frac{C_{\varepsilon_2}\tilde{\varepsilon}^2}{k}$$

- consider the log-region, where: $u^+ = \log(y^+)/\kappa + B$
- experiments show: $\mathcal{P} \approx \tilde{\varepsilon}$, and: $\langle u'v' \rangle \approx -u_\tau^2$
- with Boussinesq hypothesis, and the fact: $\mathcal{P} = -\langle u'v' \rangle \langle u' \rangle_{,y}$

$\Rightarrow$ it follows from $\tilde{\varepsilon}$-equation: $\boxed{\kappa^2 = \sigma_\varepsilon \sqrt{C_\mu}\,(C_{\varepsilon_2} - C_{\varepsilon_1})}$

$\rightarrow$ standard coefficient values yield: $\kappa = 0.43$
(compared to experimental value: $\kappa = 0.41$)

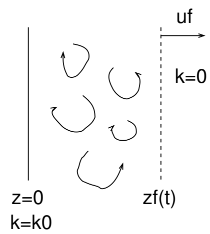
Turbulence propagating into quiescent surroundings

Consider a planar source of turbulence without mean flow

- there is no mean flow
- turbulence with $k = k_0$ and $\tilde{\varepsilon} = \tilde{\varepsilon}_0$ is emitted

$\rightarrow$ turbulence will diffuse into $z > 0$

- Lele (1985):
  diffusion velocities for k & $\tilde{\varepsilon}$ need to be equal!



$\Rightarrow$ condition leads to: $\boxed{\frac{\sigma_k}{\sigma_\varepsilon} = 6\left(\sqrt{4C_{\varepsilon 2}+1} - 2C_{\varepsilon 2}\right)}$

- link between coefficient values
  e.g. fix $C_{\varepsilon 2} = 1.92$, $\sigma_k = 1$ $\quad\Rightarrow\quad$ $\sigma_\varepsilon = 1.3$ follows
- also useful condition for Reynolds-stress transport models

Recap of the k-ε model equations

The standard k-ε model

$$\begin{aligned}
\langle u_i \rangle_{,i} &= 0 \\
\frac{\bar{\mathrm{D}}\langle u_i \rangle}{\bar{\mathrm{D}}t} &= -\left(\frac{1}{\rho}\langle p \rangle + \frac{2}{3}k\right)_{,i} + \left((\nu + \nu_T)(\langle u_i \rangle_{,j} + \langle u_j \rangle_{,i}\right)_{,j} \\
\frac{\bar{\mathrm{D}}k}{\bar{\mathrm{D}}t} &= \left(\left(\nu + \frac{\nu_T}{\sigma_k}\right) k_{,j}\right)_{,j} + \mathcal{P} - \tilde{\varepsilon} \\
\frac{\bar{\mathrm{D}}\tilde{\varepsilon}}{\bar{\mathrm{D}}t} &= \left(\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right) \tilde{\varepsilon}_{,j}\right)_{,j} + \mathcal{P}\frac{C_{\varepsilon 1}\tilde{\varepsilon}}{k} - \tilde{\varepsilon}\frac{C_{\varepsilon 2}\tilde{\varepsilon}}{k}
\end{aligned}$$

$$\nu_T = C_\mu k^2/\tilde{\varepsilon},\ C_\mu = 0.09,\ \sigma_k = 1,\ \sigma_\varepsilon = 1.3,\ C_{\varepsilon 1} = 1.44,\ C_{\varepsilon 1} = 1.92$$

- with initial and boundary conditions

→ need to prescribe values for k, $\tilde{\varepsilon}$ on all boundaries

Boundary conditions for the k-ε model equations

Laminar freestream condition

- $k = \tilde{\varepsilon} = 0$ (attention with source terms in $\tilde{\varepsilon}$ equation)

Turbulent freestream/inflow condition

- need to supply values for k, $\tilde{\varepsilon}$

⇝ often in practice: $\tilde{\varepsilon}$ not available!

Wall boundaries

- physically correct conditions: $k = 0$, $\tilde{\varepsilon} = \nu\partial_{yy}k$ (lecture 6)

⇝ but: standard model needs modifications for near-wall zone

→ more in lecture 11

The k-ω model: an alternative with two equations

Wilcox' original k-ω model equations

set $\nu_T = k/\omega$, $\tilde{\varepsilon} = C_\mu k\omega$

$$\frac{\bar{D}\langle u_i\rangle}{\bar{D}t} = -\left(\frac{1}{\rho}\langle p\rangle + \frac{2}{3}k\right)_{,i} + \left((\nu+\nu_T)(\langle u_i\rangle_{,j} + \langle u_j\rangle_{,i})\right)_{,j}$$

$$\frac{\bar{D}k}{\bar{D}t} = \left(\left(\nu + \frac{\nu_T}{\sigma_k}\right)k_{,j}\right)_{,j} + \mathcal{P} - C_\mu k\omega$$

$$\frac{\bar{D}\omega}{\bar{D}t} = \left(\left(\nu + \frac{\nu_T}{\sigma_\omega}\right)\omega_{,j}\right)_{,j} + \mathcal{P}\frac{C_{\omega_1}\omega}{k} - C_{\omega_2}\omega^2$$

- ω is a characteristic frequency of large scales
- the model is genuinely different from k-ε (cross-diffusion)
- good performance near walls; but: freestream sensitivity

Assessment of two-equation models

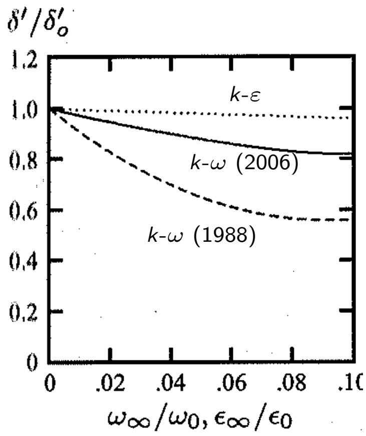
Computed spreading rate of free shear flows

	k-ε model	k-ω model	SA model	measured
plane wake	0.256	0.326	0.341	0.32-0.40
mixing layer	0.098	0.096	0.109	0.103-0.120
plane jet	0.109	0.108	0.157	0.10-0.11
round jet	0.120	0.094	0.248	0.086-0.096

(from Wilcox 2006)

High freestream sensitivity of k-ω

- free shear flow: spreading rate depends on ω_∞ (boundary cond.)
- problem related to balance between diffusion terms
- improvement in the 2006 version



(wake flow, from Wilcox 2006)

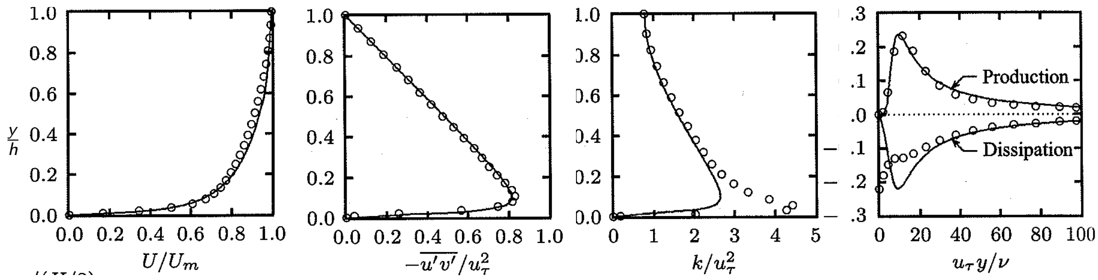
Assessment of two-equation models (2)

Channel flow predictions of the k-ω model

- mean velocity profile well predicted (c_f error $\sim 2\%$)
- but: k and ε are wrongly predicted near the wall!

$\longrightarrow$ this error *compensates* overpredicted eddy viscosity

$\Rightarrow$ result: production/shear stress is correctly captured!



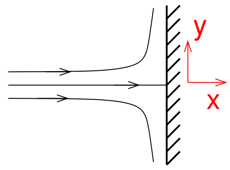
○ DNS data (Mansour et al. 1988), ——, k-ω (Wilcox 2006)

Assessment of two-equation models (3)

The stagnation point anomaly: excessive production of TKE

- consider a mean flow with stagnation point:

$\langle u\rangle = -A\,x$; $\langle v\rangle = A\,y$; $(x \le 0) \rightarrow \mathcal{P} = A(\langle u'u'\rangle - \langle v'v'\rangle)$



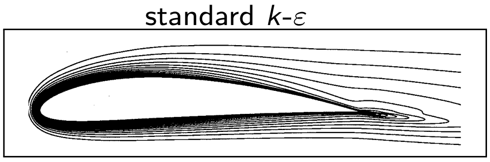
- with Boussinesq hypothesis: $\mathcal{P} = 4A^2\nu_T$

$\rightsquigarrow$ experiments & theory show much lower production ($\sim A$)

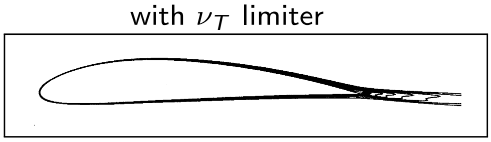
$\rightsquigarrow$ Boussinesq ansatz is not <u>realizable</u> (bounds on $\langle u_i'u_j'\rangle$)

- improvement through limiter: $\nu_T = \min(C_\mu \frac{k^2}{\varepsilon}, \frac{k}{3\lambda_{max}^{(s)}})$
- where $\lambda_{max}^{(s)}$ is the maximum eigenvalue of strain rate $\bar{S}_{ij}$

standard k-ε



with ν_T limiter



k contours in flow around airfoil (from Durbin & Petterson Reif, 2001)

Summary

k–ε model

- TKE equation as in one-equation models
- length scale constructed from dissipation rate
- transport equation for $\tilde{\varepsilon}$ difficult to model $\rightarrow$ artificial equation

$\rightsquigarrow$ standard k–ε model is not directly applicable to wall flows

k–ω model

- alternative way of determining the second scale ($\nu_t = \frac{k}{\omega}$)
- yields good results in wall-bounded flows

$\rightsquigarrow$ model is sensitive to freestream values

Summary (2)

Alleviating the deficiencies of two-equation models

1. near-wall corrections for the k–ε model (lecture 11)
2. combination of k–ε and k–ω (Menter's SST model)
3. use Reynolds-stress transport models (lecture 10)

Outlook on next lecture: Reynolds-stress transport models

How can the equations be closed at the second-moment level?

- why resort to Reynolds-stress models?
- how to derive the $\langle u_i' u_j' \rangle$ transport equation?
- how to model the principal unknown terms?

How do Reynolds-stress models perform?

Further reading

- S. Pope, *Turbulent flows*, 2000
  $\rightarrow$ chapter 10
- P.A. Durbin and B.A. Pettersson Reif, *Statistical theory and modeling for turbulent flows*, 2003
  $\rightarrow$ chapter 6
- D.C. Wilcox, *Turbulence modeling for CFD*, 2006
  $\rightarrow$ chapter 2, 3 & 4