Turbulenzmodelle in der Strömungsmechanik

Turbulent flows and their modelling

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Summary of last lecture

Lecture 4 – Free shear flows

- How does a turbulent flow develop away from solid boundaries?
- How can the equations be simplified for slow spatial evolution?
  - boundary layer approximation
- What is the evolution in the self-similar region?
  - round jet: linear spreading, mean velocities $\sim 1/x$
- Turbulence structure in the round jet:
  - turbulent kinetic energy budget
  - crude approximation with uniform turbulent viscosity
- Small scales decrease with increasing Reynolds
  - dissipation essentially independent of viscosity

LECTURE 5

The scales of turbulent motion

Questions to be answered in the present lecture

How are energy and anisotropy distributed among scales?

Which physical processes occur on each scale?

The energy cascade (Richardson 1922)

Conceptual image of energy & scales

- turbulence is composed of eddies of different sizes
- consider statistically stationary flow, very large $Re \equiv \mathcal{L}\mathcal{U}/\nu$
- characteristic size: ℓ, velocity: $u(\ell)$, timescale: $\tau(\ell) \equiv \ell/u(\ell)$
- largest eddies: $\ell = \ell_0 = \mathcal{O}(\mathcal{L})$, $u_0 \equiv u(\ell_0) = \mathcal{O}(u_{rms}) = \mathcal{O}(\mathcal{U})$
- eddies interact, transfer energy preferentially to smaller sizes
- for some size $\ell \ll \ell_0$: $Re_\ell \equiv u(\ell) \cdot \ell/\nu = \mathcal{O}(1)$

  $\longrightarrow$ dissipation by molecular viscosity becomes important

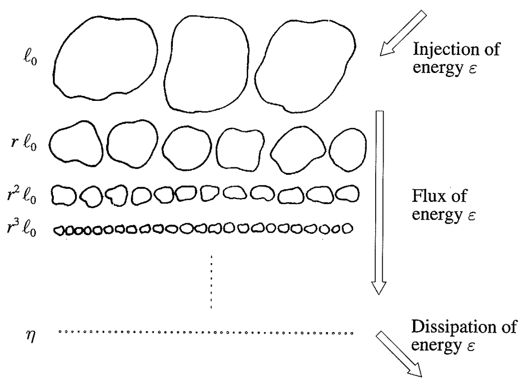
The energy cascade (2)

Consequences of the concept

- 'top-down' process
- rate of energy transfer from large scales: $u_0^2/\tau_0 = u_0^3/\ell_0$

$\longrightarrow$ dissipation scales as u_0^3/ℓ_0

$\Rightarrow$ dissipation determined by energy input!



(Frisch "Turbulence", 1995)

$\Rightarrow$ cascade process takes care of dissipating energy at the appropriate rate

Kolmogorov's theory

Quantification of the cascade

- what is the size of the smallest scales?
- how do the scales $u(\ell)$ and τ_ℓ vary along the cascade?
- how does the range of scales depend on the Reynolds number?

Kolmogorov's theory

- provides scaling laws
- provides some measurable quantities
  $\longrightarrow$ can be verified in high Reynolds number experiments
- formulated in form of hypotheses

Kolmogorov's hypotheses

Hypothesis of small-scale isotropy

> At high Reynolds numbers, the motion of small scales $\ell \ll \ell_0$ is statistically isotropic.

- directional bias & information about flow geometry
  $\longrightarrow$ lost along the cascade

$\Rightarrow$ small-scale statistics should be universal

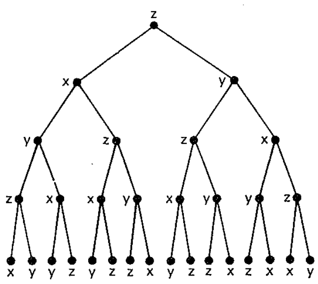
Hypothesis 1: Small-scale isotropy

Loss of anisotropy due to repeated vortex stretching

Cartoon-like explanation

- vortex "stretching" term: $(\omega \cdot \nabla)\mathbf{u}$
- stretching in z $\rightarrow$ gradients in x,y
- and so on ...

$\Rightarrow$ isotropization after repeated steps



Frequency of symbols at each generation

x	y	z
0	0	1
1	1	0
1	1	2
3	3	2
5	5	6
11	11	10
21	21	22

(Bradshaw 1971)

Hypothesis 2: Similarity of small scales

First similarity hypothesis

At high Reynolds numbers, the statistics of the small-scale motion ($\ell < \ell_{EI}$) have a universal form determined by ν and ε.

- Kolmogorov scales (from dimensional grounds):

$$\eta \equiv \left(\nu^3/\varepsilon\right)^{1/4}, \quad \tau_\eta \equiv (\nu/\varepsilon)^{1/2}, \quad u_\eta \equiv (\nu\varepsilon)^{1/4}$$

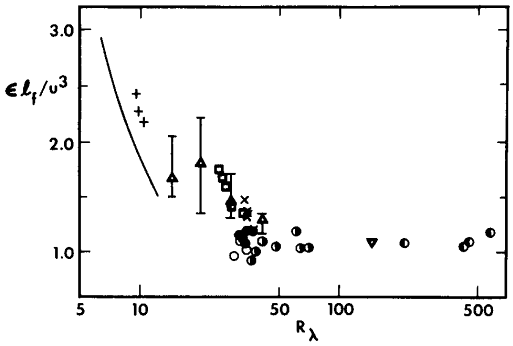
$\Rightarrow$ recall: $Re_\eta \equiv \eta\, u_\eta/\nu = 1 \rightarrow$ viscous effects important!

- scales decrease with large-scale Reynolds number: $\eta/\ell_0 \sim Re^{-3/4}, \quad u_\eta/u_0 \sim Re^{-1/4}, \quad \tau_\eta/\tau_0 \sim Re^{-1/2}$
- η decreases faster than $u_\eta \rightarrow$ gradients increase

Finite limit for dissipation at high *Re*

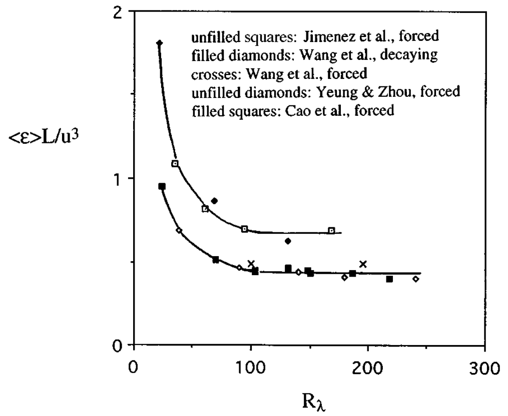
Experimental evidence from homogeneous-isotropic turbulence

laboratory grid turbulence



(Sreenivasan 1984)

direct numerical simulation



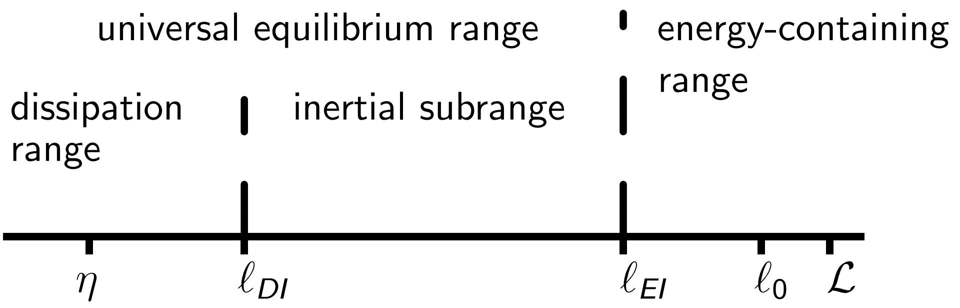
(Sreenivasan 1998)

$\Rightarrow \varepsilon\ell/u_0^3$ has finite value $\mathcal{O}(1)$

Hypothesis 3: Inertial similarity

Second similarity hypothesis

At high Reynolds numbers, the statistics of motions in the range $\ell_0 \gg \ell \gg \eta$ have a universal form determined by ℓ and ε, independent of ν.



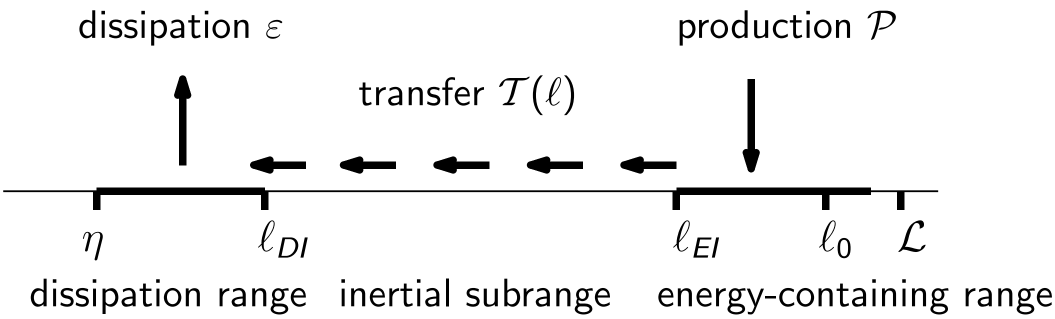
- inertial subrange scales: $u(\ell) = (\varepsilon\ell)^{1/3}$, $\tau_\ell = (\ell^2/\varepsilon)^{1/3}$

Energy flux through the cascade

- rate of energy transfer through scale ℓ defined as $\mathcal{T}(\ell)$
- determined by scales around ℓ: $\quad \mathcal{T}(\ell) \sim u(\ell)^2/\tau_\ell = \varepsilon$

$\Rightarrow$ transfer rate $\mathcal{T}(\ell)$ is independent of ℓ !

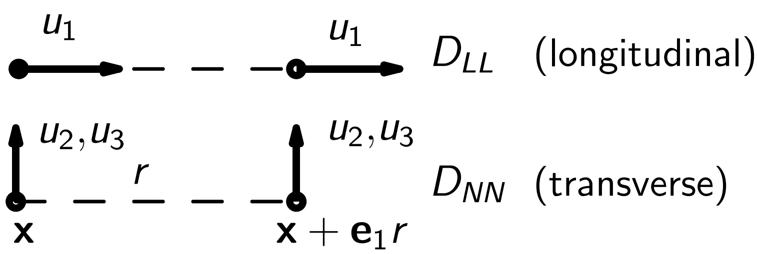
- conceptual diagram of the cascade:



Predictions by Kolmogorov's theory

Second order velocity structure function (definitions)

- $D_{ij}(\mathbf{r},\mathbf{x},t) \equiv \langle (u_i(\mathbf{x}+\mathbf{r},t) - u_i(\mathbf{x},t))\,(u_j(\mathbf{x}+\mathbf{r},t) - u_j(\mathbf{x},t))\rangle$
- related to: $R_{ij}(\mathbf{r},\mathbf{x},t) = \langle u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)\rangle$ (lecture 3)
- local isotropy: for $|\mathbf{r}| \ll \mathcal{L}$ only components D_{LL}, D_{NN}

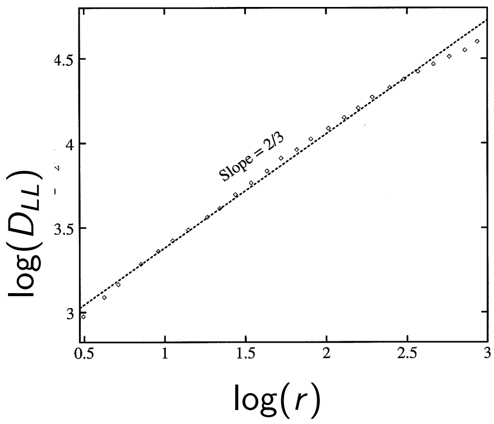


- in homogeneous turbulence with $\langle u_i \rangle = 0$
  from continuity: $D_{NN} = D_{LL} + \frac{1}{2} r\partial_r D_{LL}$

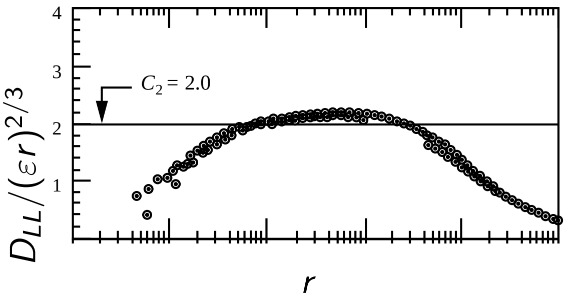
$\Rightarrow$ in this case: $D_{ij}(\mathbf{r},t)$ fully determined by $D_{LL}(r,t)$

Second order velocity structure function (K41 results)

- second similarity hypothesis (inertial subrange):
  statistics depend only on r, ε $\quad \boxed{D_{LL} = C_2\,(\varepsilon r)^{2/3}}$



(grid turbulence, Gagne & Hopfinger)



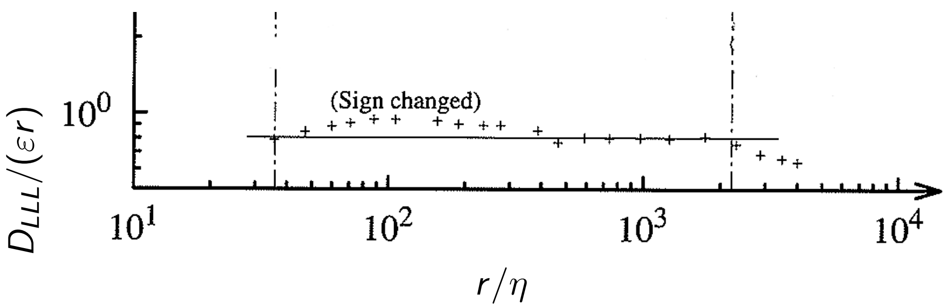
(boundary layer, Saddoughi & Veeravalli 1994)

K41 results: the 4/5 law for isotropic turbulence

Result derived from Navier-Stokes

- derive transport equation for D_{LL}
  (involves 3rd order structure function D_{LLL})
- invoke: local isotropy, negligible viscosity (inertial subrange)

$\Rightarrow$ Kolmogorov's 4/5-law: $\quad \boxed{D_{LLL}(r) = -\frac{4}{5}\,\varepsilon r}$



(grid turbulence, Gagne 1987)

Spectral view of the cascade

Previous arguments were based on physical space view

Alternative – spectral space view:

- based upon Fourier transform

1. introduce spectral quantities
2. present consequences of Kolmogorov's theory
3. discuss energy cascade in wavenumber space

Velocity spectrum tensor

Homogeneous turbulence

- definition: (cf. lecture 3)
  spectrum tensor Φ_{ij} = transform of two-point correlation R_{ij}

$$\begin{aligned}\Phi_{ij}(\boldsymbol{\kappa}, t) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{-I\boldsymbol{\kappa}\cdot\mathbf{r}} R_{ij}(\mathbf{r}, t)\,\mathrm{d}\mathbf{r} \\ R_{ij}(\mathbf{r}, t) &= \int_{-\infty}^{\infty} e^{+I\boldsymbol{\kappa}\cdot\mathbf{r}} \Phi_{ij}(\boldsymbol{\kappa}, t)\,\mathrm{d}\boldsymbol{\kappa}\end{aligned}$$

- setting $\mathbf{r} = 0$: $R_{ij}(0, t) = \langle u_i' u_j' \rangle = \int_{-\infty}^{\infty} \Phi_{ij}(\boldsymbol{\kappa}, t)\,\mathrm{d}\boldsymbol{\kappa}$

$\Rightarrow$ $\Phi_{ij}(\boldsymbol{\kappa})$ is contribution from mode $\boldsymbol{\kappa}$ to Reynolds stress

Energy spectrum function

Reduction of information contained in Φ_{ij}

- sum diagonal components, integrate over directions of $\boldsymbol{\kappa}$:

$$E(\kappa, t) = \int_{-\infty}^{\infty} \frac{1}{2} \Phi_{ii}(\boldsymbol{\kappa}, t) \delta(|\boldsymbol{\kappa}| - \kappa) \, \mathrm{d}\boldsymbol{\kappa}$$

- $E(\kappa)$ is real, non-negative, defined for $\kappa \geq 0$
- $k = \int_0^{\infty} E(\kappa) \, \mathrm{d}\kappa$ contribution to TKE
- $\varepsilon = \int_0^{\infty} 2\nu\kappa^2 E(\kappa) \, \mathrm{d}\kappa$ contribution to dissipation

Kolmogorov spectra

Scaling in the inertial subrange ($\eta \ll \ell \ll \ell_0$)

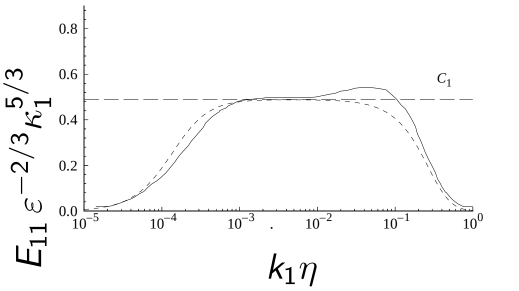
- recall the 2/3 law: $D_{LL} = C_2 \, (\varepsilon r)^{2/3}$
- it is possible to relate $D_{LL}(r)$ to spectrum function $E(\kappa)$

$\Rightarrow$ $\boxed{E(\kappa) = C_{kol} \, \varepsilon^{2/3} \kappa^{-5/3}}$

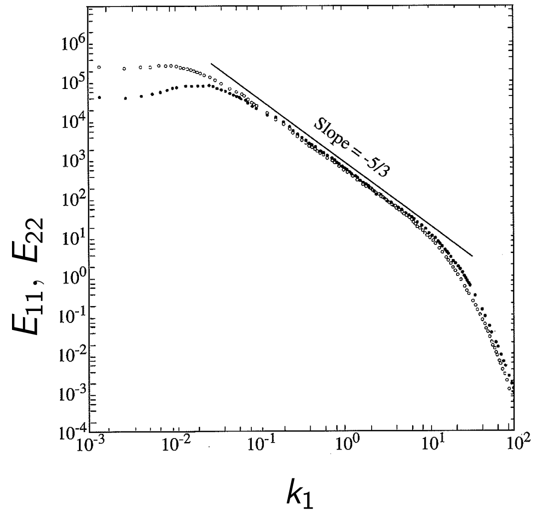
- universal constant: $C_{kol} = 1.5$
  (directly related to C_2, value from measurements)
- confirmed in numerous experiments at high Reynolds number

Kolmogorov spectra - experimental confirmation

1D energy spectra – same scaling as corresponding 3D spectra



(boundary layer, Saddoughi & Veeravalli 1994)

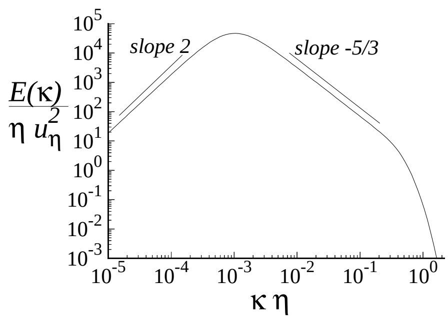


(high Re jet flow, Champagne 1978)

A model spectrum

Pope (2000) proposes models spectrum

- valid over range of scales: $E(\kappa) = \underbrace{C_{kol}\,\varepsilon^{2/3}\kappa^{-5/3}}_{\text{Kolmogorov}} f_L(\kappa L)\, f_\eta(\kappa\eta)$
- $f_L(\kappa L)$ – energy-containing range
  - tends to unity for large κL
  - for small κL: $E(\kappa) \sim k^{p_0}$
- $f_\eta(\kappa\eta)$ – dissipation range:
  - tends to unity for small $\kappa\eta$
  - for large $\kappa\eta$: $E(\kappa) \sim \exp(-\beta\kappa\eta)$



given Re, $p_0 = 2$

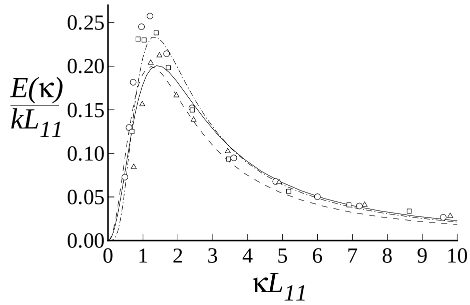
Spectral behaviour of the large scales

Energy-containing range

- non-universal behavior!
- 3D spectrum function more informative than 1D (aliasing)

⇒ consider grid turbulence
→ approx. isotropic

- $$\int_0^\infty E(\kappa)/\kappa \, d\kappa = \frac{4}{3\pi} k L_{11}$$



○ Comte-Bellot & Corrsin 1971, $Re_\lambda = 60 \ldots 70$

—— model spectrum, $Re_\lambda = 60$, $p_0 = 2$

- - - - model spectrum, $Re_\lambda = 1000$, $p_0 = 2$

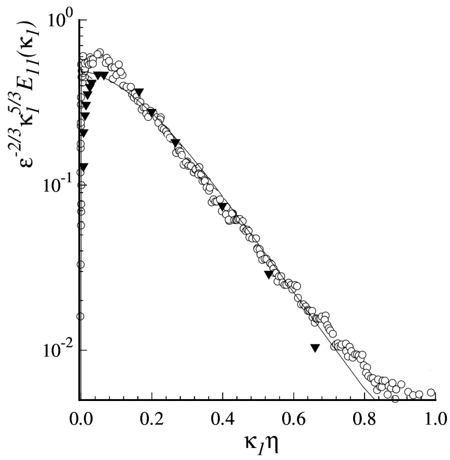
—·— model spectrum, $Re_\lambda = 60$, $p_0 = 4$

The Richardson cascade
Kolmogorov's 1941 theory
Spectral view
Velocity spectra
Spectrum balance
Summary

Spectral behaviour of the dissipation range

Dissipation range

- universal for different flows
- lin-log plot:
  straight = exponential decay
- peak dissipation at $\ell/\eta \approx 24$



▼ grid-turbulence Comte-Bellot & Corrsin 1971, $Re_\lambda = 60$

○ boundary layer, Saddoughi & Veeravalli 1994, $Re_\lambda = 600$

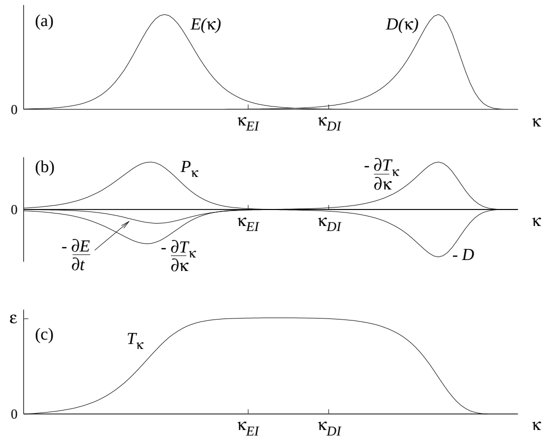
—— model spectrum, $Re_\lambda = 600$

Energy spectrum balance in homogeneous turbulence

$$\partial_t E(\kappa) = \underbrace{\mathcal{P}_\kappa(\kappa,t)}_{\text{production}} - \underbrace{\partial_\kappa \mathcal{T}_\kappa(\kappa,t)}_{\text{spectral transfer}} \underbrace{-2\nu\kappa^2 E(\kappa,t)}_{\text{dissipation } \equiv \mathcal{D}(\kappa)}$$

- production limited to energy-containing range
- $\kappa < \kappa_{EI}$:
  $\partial_t E = \mathcal{P}_\kappa - \partial_\kappa \mathcal{T}_\kappa$
- $\kappa_{EI} < \kappa < \kappa_{DI}$:
  $0 = -\partial_\kappa \mathcal{T}_\kappa$
- $\kappa_{DI} < \kappa$:
  $0 = -\partial_\kappa \mathcal{T}_\kappa - \mathcal{D}$

sketch for high Reynolds no. flows



(Pope "Turbulence', 2000)

Summary of the lecture

The turbulent energy cascade

- hierarchy of eddies, downward transfer of energy
- dissipation determined by large scales, performed by small scales

Kolmogorov's theory

- building block of turbulence research
- valuable results for small scales (e.g. $-5/3$ spectrum)

BUT: Problem of non-universality of large scales remains

Outlook on next lecture: Wall turbulence

What is the general structure of wall-bounded flows?

How does the presence of a solid boundary affect the turbulent motion?

What is the effect of wall roughness?

Further reading

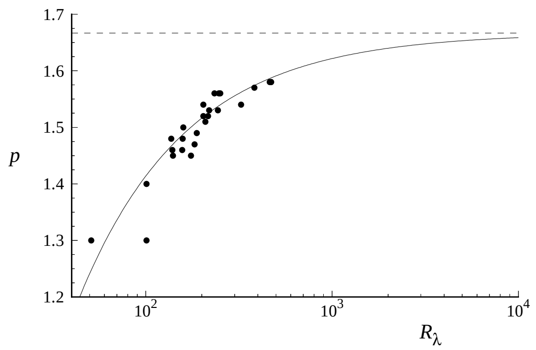
- S. Pope, *Turbulent flows*, 2000
  $\longrightarrow$ chapter 6
- U. Frisch, *Turbulence*, 1995
  $\longrightarrow$ chapters 6-8

Appendix

Shortcomings and refinements

Reynolds number dependence

spectral exponent in inertial subrange



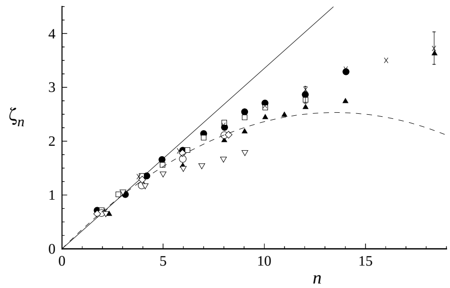
(Mydlarski & Warhaft 1998)

- define $E(\kappa) \sim \kappa^{-p}$
- exponent p approaches $5/3$ only slowly with Reynolds

Shortcomings and refinements (2)

Higher-order statistics deviate from K41 theory

- nth order structure function
  $D_n(r) \equiv \langle (\Delta_r u)^n \rangle$
- K41, dimensional arguments:
  $D_n(r) \sim (\varepsilon r)^{\zeta_n}$ $\qquad \zeta_m = n/3$
- $n > 3$: measurements deviate
  $\rightarrow$ attributed to intermittency
- refinements by Kolmogorov (1962)



● Anselmet et al. 1984

—— K41

---- refined similarity