

Turbulenzmodelle in der Strömungsmechanik

Turbulent flows and their modelling

Markus Uhlmann

Institut für Hydromechanik

www.ifh.uni-karlsruhe.de/people/uhlmann

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Summary of last lecture

Lecture 3 – Statistical description of turbulence

- ▶ How do we compute and analyze a turbulent flow statistically?
- ▶ Part I: What are the basic mathematical tools?
 - ▶ random variables, processes, fields
 - ▶ two-time/two-point correlations
 - ▶ Fourier space analysis
- ▶ Part II: What are the averaged equations?
 - ▶ Reynolds decomposition and averaging
 - ▶ closure problem, turbulent viscosity

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LECTURE 4

Free shear flows

Questions to be answered in the present lecture

How does a turbulent flow develop away from solid boundaries?

How can the equations be simplified for slow spatial evolution?

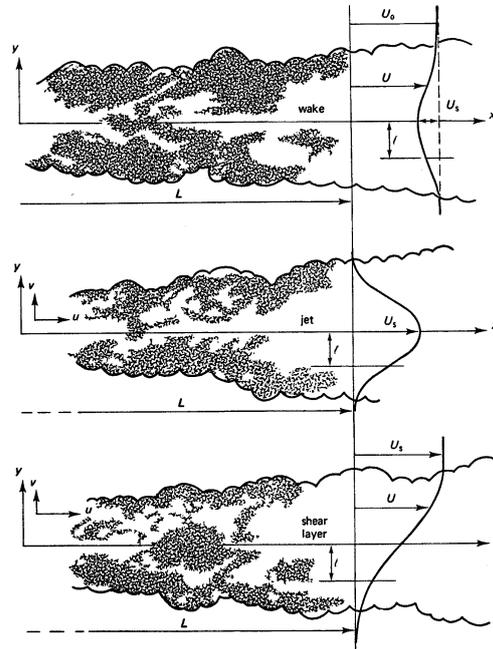
What is the evolution in the self-similar region?

What is the turbulence structure in a plane jet?

Types of free shear flows

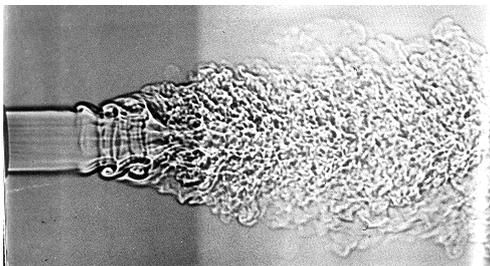
Flows developing far from solid boundaries

- ▶ wake
- ▶ jet
- ▶ mixing layer

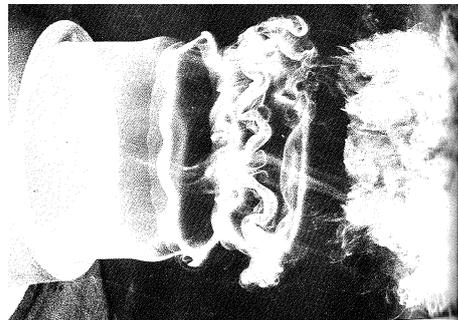


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The round jet



$Re = 3000$



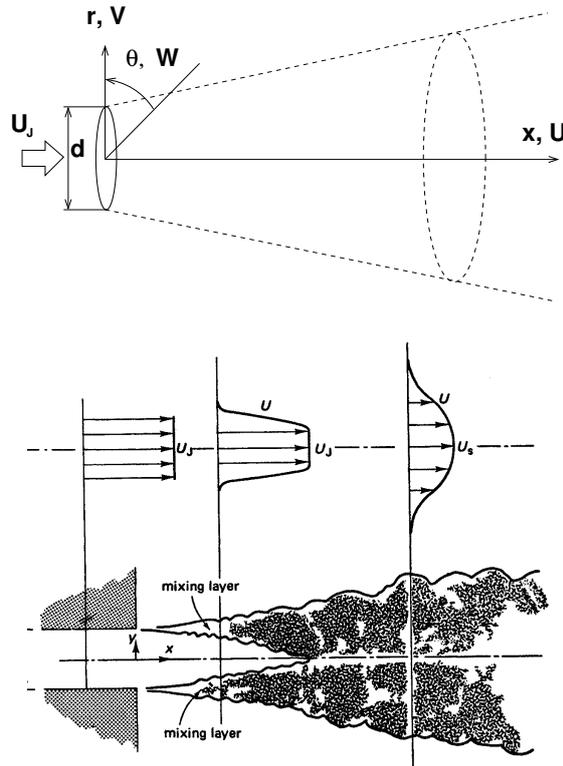
$Re = 13000$

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Configuration and coordinate system

Steady round jet

- ▶ quiescent surroundings
- ▶ nozzle diameter d
- ▶ exit velocity U_J
- Reynolds: $Re = U_J d / \nu$
- ▶ statistically stationary & axisymmetric flow
- ▶ initial development ($20d$) then self-similar

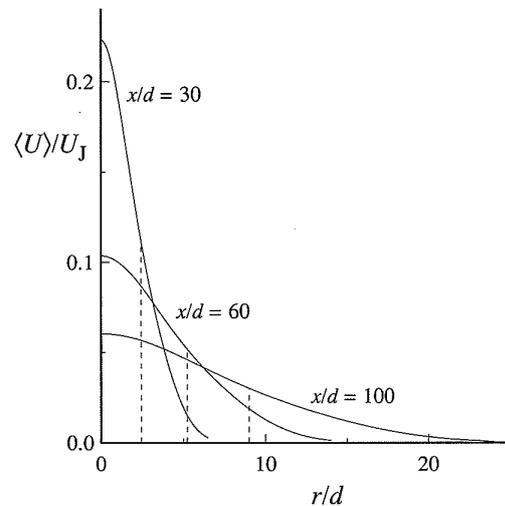


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Mean velocity and self-similarity

Observations from experiments

- ▶ mean axial velocity $\langle u(x, r) \rangle$
- velocity decays on axis & jet spreads radially
- ▶ profile shape remains similar
- ▶ define centerline velocity: $U_0(x) \equiv \langle u(x, 0) \rangle$
- ▶ define jet half width $r_{1/2}(x)$: $\langle u(x, r_{1/2}) \rangle = U_0(x)/2$



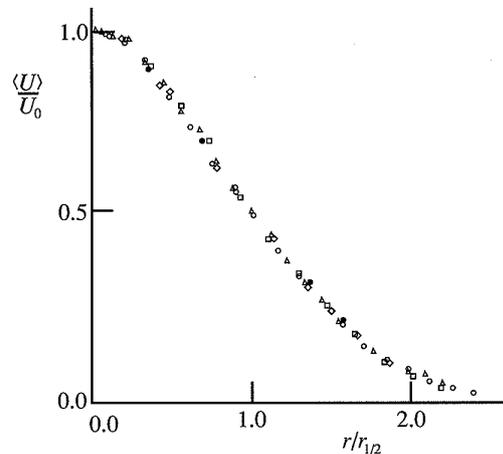
($Re = 95500$, Hussein et al. 1994)

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Mean velocity and self-similarity (2)

Observations from experiments

- ▶ scaling of radial coordinate:
 $\xi \equiv r/r_{1/2}$
- ▶ scaling of velocity:
 $f(\xi) \equiv \langle u(x, r) \rangle / U_0$
- profiles collapse
- ▶ spreading rate S :
 $r_{1/2}(x) = S(x - x_0)$
- ▶ velocity decay:
 $U_0(x) = B \cdot d \cdot U_J / (x - x_0)$
- local Reynolds is constant!
 $Re_0(x) = r_{1/2} U_0 / \nu$



($Re = 10^5$, Wygnanski & Fiedler 1969)

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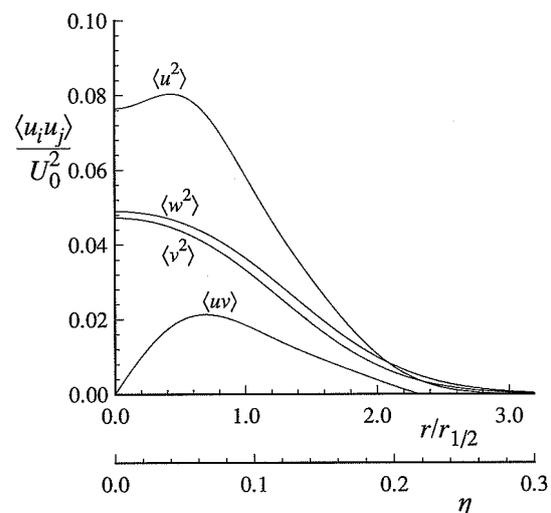
Reynolds stresses

Observations from experiments

- ▶ coordinate system:
 $\mathbf{x} = (x, r, \theta)$, $\mathbf{u} = (u, v, w)$
- ▶ due to symmetry:
 $\langle w \rangle = \langle u' w' \rangle = \langle v' w' \rangle = 0$

$$\begin{bmatrix} \langle u' u' \rangle & \langle u' v' \rangle & 0 \\ \langle u' v' \rangle & \langle v' v' \rangle & 0 \\ 0 & 0 & \langle w' w' \rangle \end{bmatrix}$$

- ▶ self-similarity of $\langle u'_i u'_j \rangle / U_0^2$
- ▶ significant anisotropy



(Hussein et al. 1994)

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Reynolds stresses

Observations from experiments

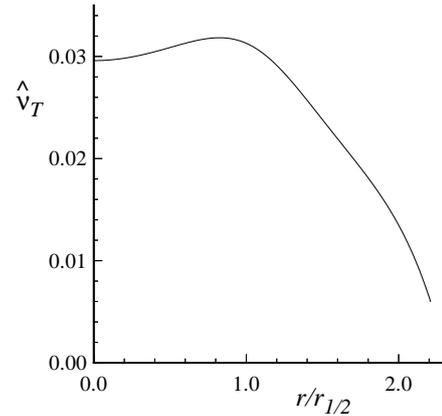
- ▶ shear stress can be written:

$$\langle u'v' \rangle = -\nu_T \frac{\partial \langle u \rangle}{\partial r}$$

- ▶ positive eddy viscosity ν_T
- ▶ self-similarity:

$$\hat{\nu}_T(r) \equiv \frac{\nu_T(x, r)}{U_0(x) r_{1/2}(x)}$$

⇒ plateau across jet



(Hussein et al. 1994)

Navier-Stokes equations in cylindrical coordinates

Instantaneous equations

$$\partial_x u + \frac{1}{r} \partial_r (rv) + \frac{1}{r} \partial_\theta w = 0$$

$$\partial_t u + u \partial_x u + v \partial_r u + \frac{w}{r} \partial_\theta u = -\frac{1}{\rho} \partial_x p + \nu \nabla^2 u$$

$$\partial_t v + u \partial_x v + v \partial_r v + \frac{w}{r} \partial_\theta v - \frac{w^2}{r} = -\frac{1}{\rho} \partial_r p + \nu \left(\nabla^2 v - \frac{v}{r^2} - \frac{2}{r^2} \partial_\theta w \right)$$

$$\partial_t w + u \partial_x w + v \partial_r w + \frac{w}{r} \partial_\theta w + \frac{vw}{r} = -\frac{1}{r\rho} \partial_\theta p + \nu \left(\nabla^2 w - \frac{w}{r^2} + \frac{2}{r^2} \partial_\theta v \right)$$

with:

$$\nabla^2 = \partial_{xx} + \frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \partial_{\theta\theta}$$

Reynolds-averaged Navier-Stokes in cylindrical coordinates

Averaged equations

- ▶ non-swirling, statistically stationary & axisymmetric

$$0 = \partial_x \langle u \rangle + \frac{1}{r} \partial_r (r \langle v \rangle)$$

$$\frac{\bar{D} \langle u \rangle}{\bar{D} t} = -\frac{1}{\rho} \partial_x \langle p \rangle - \partial_x \langle u' u' \rangle - \frac{1}{r} \partial_r (r \langle u' v' \rangle) + \nu \nabla^2 \langle u \rangle$$

$$\begin{aligned} \frac{\bar{D} \langle v \rangle}{\bar{D} t} = & -\frac{1}{\rho} \partial_r \langle p \rangle - \partial_x \langle u' v' \rangle - \frac{1}{r} \partial_r (r \langle v' v' \rangle) + \frac{\langle w' w' \rangle}{r} \\ & + \nu \left(\nabla^2 \langle v \rangle - \frac{\langle v \rangle}{r^2} \right) \end{aligned}$$

$$\text{with: } \frac{\bar{D}}{\bar{D} t} = \partial_t + \langle u \rangle \partial_x + \langle v \rangle \partial_r$$

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Boundary layer approximation

Applicable to flows with slow spatial evolution

- ▶ small streamwise gradients: $\partial_x \sim 1/L$
 - ▶ large cross-stream variation: $\partial_r \sim 1/\delta$
 with $L \gg \delta$
- ⇒ simplification of Navier-Stokes equations

Examples:

- ▶ free shear flows (jet, wake, mixing layer)
- ▶ wall-bounded flows (boundary layer)

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Boundary layer equations for round jet

Perform order-of-magnitude analysis

- ▶ U, V are reference velocities
- ▶ continuity equation:

$$\underbrace{\partial_x \langle u \rangle}_{\mathcal{O}\left(\frac{U}{L}\right)} + \underbrace{\frac{1}{r} \partial_r (r \langle v \rangle)}_{\mathcal{O}\left(\frac{V}{\delta}\right)} = 0$$

- ▶ balance of the terms:

$$V \sim \frac{U\delta}{L}$$

⇒ reference velocity in y-direction: $\frac{U\delta}{L}$

Boundary layer equations for round jet (2)

Radial momentum equation

term	order $\mathcal{O}(\cdot)$	$\mathcal{O}(\cdot)$, divided by $U^2\delta/L^2$	dominant
$\partial_t \langle v \rangle$	0 (stationary)	0	
$+\langle u \rangle \partial_x \langle v \rangle$	$UV/L = U^2\delta/L^2$	1	
$+\langle v \rangle \partial_r \langle v \rangle$	$V^2/\delta = U^2\delta/L^2$	1	
$= -\frac{1}{\rho} \partial_r \langle p \rangle$	$\Delta p_r / (\rho\delta)$	$\Delta p_r L^2 / (\rho U^2 \delta^2)$	•
$-\partial_x \langle u'v' \rangle$	R_{12}/L	$R_{12}L / (U^2\delta)$	
$-\frac{1}{r} \partial_r (r \langle v'v' \rangle)$	R_{22}/δ	$R_{22}L^2 / (U^2\delta^2)$	•
$+\frac{1}{r} \langle w'w' \rangle$	R_{33}/δ	$R_{33}L^2 / (U^2\delta^2)$	•
$+\nu \partial_{xx} \langle v \rangle$	$\nu V/L^2 = \nu U\delta/L^3$	$1/Re$	
$+\nu \frac{1}{r} \partial_r (r \partial_r \langle v \rangle)$	$\nu V/\delta^2 = \nu U/(\delta L)$	$\frac{1}{Re} \frac{L^2}{\delta^2}$	
$+\nu \frac{1}{r^2} \partial_{\theta\theta} \langle v \rangle$	0 (axisymmetric)	0	
$-\nu \frac{\langle v \rangle}{r^2}$	$\nu V/\delta^2 = \nu U/(L\delta)$	$\frac{1}{Re} \frac{L^2}{\delta^2}$	

$$\Rightarrow \frac{1}{\rho} \partial_r \langle p \rangle + \frac{1}{r} \partial_r (r \langle v'v' \rangle) - \frac{1}{r} \langle w'w' \rangle = 0$$

Boundary layer equations for round jet (3)

Obtaining the streamwise pressure gradient

- ▶ integrate the approximate radial momentum equation:

$$\frac{\langle p \rangle}{\rho} = \frac{p_0(x)}{\rho} - \langle v'v' \rangle + \int_r^\infty \left(\frac{\langle v'v' \rangle}{r'} - \frac{\langle w'w' \rangle}{r'} \right) dr'$$

- ▶

$$\frac{1}{\rho} \partial_x \langle p \rangle = \frac{1}{\rho} \frac{dp_0}{dx} - \partial_x \langle v'v' \rangle + \partial_x \int_r^\infty \left(\frac{\langle v'v' \rangle}{r'} - \frac{\langle w'w' \rangle}{r'} \right) dr'$$

- ▶ neglecting streamwise gradients of Reynolds stresses:

$$\partial_x \langle p \rangle \approx \frac{dp_0}{dx}$$

Boundary layer equations for round jet (4)

Streamwise momentum equation

term	order $\mathcal{O}(\cdot)$	$\mathcal{O}(\cdot)$, divided by U^2/L	dominant
$\partial_t \langle u \rangle$	0 (stationary)	0	
$+\langle u \rangle \partial_x \langle u \rangle$	U^2/L	1	•
$+\langle v \rangle \partial_r \langle u \rangle$	$VU/\delta = U^2/L$	1	•
$= -\frac{1}{\rho} \partial_x \langle p \rangle$	$\Delta p_x/(\rho L)$	$\Delta p_x/(\rho U^2)$	•
$-\partial_x \langle u'u' \rangle$	R_{11}/L	R_{11}/U^2	
$-\frac{1}{r} \partial_r (r \langle u'v' \rangle)$	R_{12}/δ	$R_{12}L/(U^2\delta)$	•
$+\nu \partial_{xx} \langle u \rangle$	$\nu U/L^2$	$1/Re$	
$+ \nu \frac{1}{r} \partial_r (r \partial_r \langle u \rangle)$	$\nu U/\delta^2$	$\frac{1}{Re\delta^2}$	
$+ \nu \frac{1}{r^2} \partial_{\theta\theta} \langle u \rangle$	0 (axisymmetric)	0	

$$\Rightarrow \langle u \rangle \partial_x \langle u \rangle + \langle v \rangle \partial_r \langle u \rangle = -\frac{1}{\rho} \partial_x \langle p \rangle - \frac{1}{r} \partial_r (r \langle u'v' \rangle)$$

Boundary layer equations for round jet (5)

$$\partial_x \langle u \rangle + \frac{1}{r} \partial_r (r \langle v \rangle) = 0$$

$$\langle u \rangle \partial_x \langle u \rangle + \langle v \rangle \partial_r \langle u \rangle = -\frac{1}{\rho} \frac{d\rho}{dx} - \frac{1}{r} \partial_r (r \langle u'v' \rangle)$$

Using the BL approximation:

- ▶ it can be shown that self-similarity implies:

$$U_0 \sim \frac{1}{(x - x_0)}$$

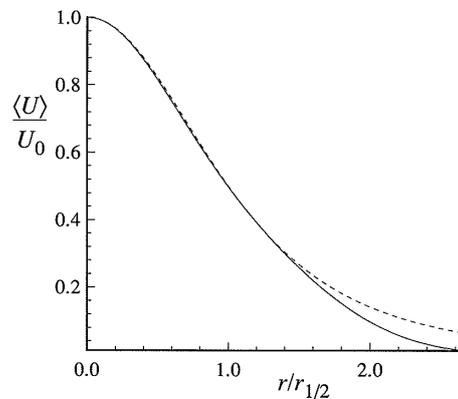
$$r_{1/2} \sim x - x_0$$

→ consistent with experimental observations (cf. above)

Closure of equations in BL approximation

Assuming a uniform turbulent viscosity

- ▶ define $-\langle u'v' \rangle = \nu_T \partial_r \langle u \rangle$
- ▶ experiments show similarity:
 $\nu_T(x, r) = r_{1/2}(x) U_0(x) \hat{\nu}_T(r)$
- ▶ $\hat{\nu}_T \approx 0.028$ across jet
- ⇒ assume $\hat{\nu}_T = cst.$
- ▶ BL equations closed
- ⇒ reasonable prediction of $\langle u \rangle$
 (disagreement near edge)



$\nu_T = cst$ approx. vs. Hussein et al. 1994

Kinetic energy equation

Instantaneous kinetic energy

- ▶ definition: $E_k(\mathbf{x}, t) \equiv \frac{1}{2} \mathbf{u} \cdot \mathbf{u}$
- ▶ transport equation (cf. lecture 2):

$$\frac{DE_k}{Dt} = - (u_j p / \rho)_{,j} + (2\nu u_i S_{ij})_{,j} - 2\nu S_{ij} S_{ij}$$

Mean kinetic energy

- ▶ decomposition:

$$\langle E \rangle = \underbrace{\frac{1}{2} \langle \mathbf{u} \cdot \mathbf{u} \rangle}_{\equiv \bar{E}} + \underbrace{\frac{1}{2} \langle \mathbf{u}' \cdot \mathbf{u}' \rangle}_{\equiv k}$$

(due to mean flow) (due to turbulence)

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Kinetic energy of mean flow and turbulence

Transport equations

$$\begin{aligned} \partial_t \bar{E} + (\langle u_j \rangle \bar{E} + \langle u_i \rangle \langle u'_i u'_j \rangle + \langle u_j \rangle \langle p \rangle / \rho - 2\nu \langle u_i \rangle \bar{S}_{ij})_{,j} &= -\mathcal{P} - \bar{\varepsilon} \\ \partial_t k + \left(\langle u_j \rangle k + \frac{1}{2} \langle u'_i u'_i u'_j \rangle + \langle u'_j p' \rangle / \rho - 2\nu \langle u'_i S'_{ij} \rangle \right)_{,j} &= +\mathcal{P} - \varepsilon \end{aligned}$$

- ▶ production term: $\mathcal{P} \equiv -\langle u'_i u'_j \rangle \langle u_i \rangle_{,j}$
- ⇒ sink for \bar{E} , source for k → exchange term!
- ▶ dissipation due to mean flow: $\bar{\varepsilon} \equiv 2\nu \bar{S}_{ij} \bar{S}_{ij}$
- ▶ dissipation due to turbulence: $\varepsilon \equiv 2\nu \langle S'_{ij} S'_{ij} \rangle$

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Scaling of dissipation

Round jet flow

- ▶ considering self-similarity of $\langle u \rangle$, $\langle u'_i u'_j \rangle$
- ▶ production term (BL approximation) scales as:

$$\mathcal{P}/(U_0^3/r_{1/2}) \approx -\frac{\langle u'v' \rangle}{U_0^2} \frac{r_{1/2}}{U_0} \partial_r \langle u \rangle$$

→ turbulent dissipation will also scale as:

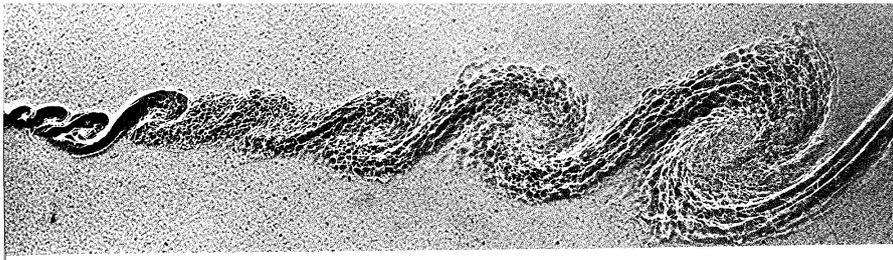
$$\hat{\varepsilon} = \varepsilon/(U_0^3/r_{1/2})$$

⇒ dissipation defined as $2\nu \langle s'_{ij} s'_{ij} \rangle$, but found independent of ν !

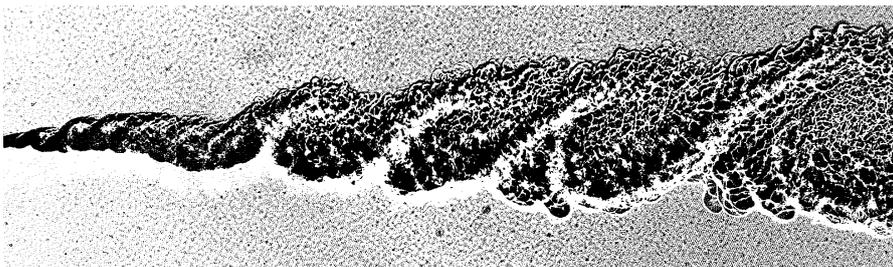
- ▶ finer scales at higher Re → higher gradients s'_{ij}

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Finer scales with increasing Reynolds



$Re = 10^5$



$Re = 2 \cdot 10^5$

(Brown & Roshko 1974)

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Kolmogorov scales

Why is dissipation independent of the value for viscosity?

- ▶ define characteristic scales of smallest turbulent motion:

$$\eta \equiv (\nu^3/\varepsilon)^{1/4}, \quad \tau_\eta \equiv (\nu/\varepsilon)^{1/2}, \quad u_\eta \equiv (\nu\varepsilon)^{1/4}$$

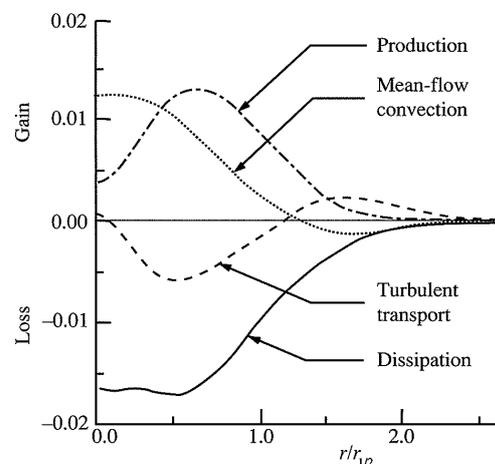
- ▶ Kolmogorov-scale Reynolds number: $Re_\eta \equiv \eta u_\eta/\nu = 1$
- ▶ length scale decreases with Reynolds: $\eta/r_{1/2} = Re_0^{-3/4} \hat{\varepsilon}^{-1/4}$
→ compensates changes in viscosity
- ▶ more details on scaling in lecture 5

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Turbulent kinetic energy budget

Observations from experiments

- ▶ mean-flow convection:
 - $(\langle u_j \rangle k)_j$
- ▶ turbulent transport:
 - $\left(\frac{1}{2} \langle u'_i u'_i u'_j \rangle + \langle u'_j p' \rangle / \rho - 2\nu \langle u'_i S'_{ij} \rangle \right)_j$
- ▶ production: \mathcal{P}
- ▶ dissipation: $-\varepsilon$



(Panchapakesan & Lumley 1993)

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Comparison of time scales in turbulent free shear flow

Round jet data

- ▶ reference scale:

$$\tau_0 \equiv r_{1/2}/U_0$$

$$\tau_0/\tau_0 = 1$$

- ▶ mean shear:

$$\tau_S \equiv 1/\partial_r \langle U \rangle$$

$$\tau_S/\tau_0 \approx 1.7$$

- ▶ turbulence production:

$$\tau_P \equiv k/\mathcal{P}$$

$$\tau_P/\tau_0 \approx 6$$

- ▶ turbulence dissipation:

$$\tau_\varepsilon \equiv k/\varepsilon$$

$$\tau_\varepsilon/\tau_0 \approx 4.5$$

- ▶ mean “flight time” from virtual origin:

$$\tau_J \equiv \frac{1}{2}x/U_0$$

$$\tau_J/\tau_0 \approx 5.3$$

→ turbulence is long-lived

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Other canonical free shear flows

Plane jet

Mixing layer

Plane or axisymmetric wake

Homogeneous shear flow

Homogeneous-isotropic flow

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Summary

Main questions of the present lecture

- ▶ How does a turbulent flow develop away from solid boundaries?
- ▶ How can the equations be simplified for slow spatial evolution?
 - ▶ boundary layer approximation
- ▶ What is the evolution in the self-similar region?
 - ▶ round jet: linear spreading, mean velocities $\sim 1/x$
- ▶ Turbulence structure in the round jet:
 - ▶ turbulent kinetic energy budget
 - ▶ crude approximation with uniform turbulent viscosity

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Problem

Derive the formula for the mean “flight time” of a fluid particle on the round jet axis, measured from the virtual origin x_0 to a position x . (Answer: $\tau_J = \frac{1}{2}x/U_0$)

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Outlook on next lecture: The scales of turbulence

How are energy and anisotropy distributed among scales?

Which physical processes occur on each scale?

Further reading

- ▶ S. Pope, *Turbulent flows*, 2000
→ chapter 5
- ▶ H. Tennekes and J.L. Lumley, *First Course in Turbulence*, 1972
→ chapter 5