Turbulenzmodelle in der Strömungsmechanik

Turbulent flows and their modelling

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Schedule

1	General introduction to turbulent flows	20.10.
2	Equations of fluid motion	27.10.
3	Statistical description of turbulence	10.11.
4	Free shear flows	17.11.
5	The scales of turbulent motion	24.11.
6	Wall-bounded shear flows	1.12.
7	DNS as numerical experiments	8.12.
8	Introduction to RANS modelling	15.12.
9	k–ε and other eddy viscosity models	12.1.
10	Reynolds-stress transport models	19.1.
11	Boundary conditions and wall treatment	26.1.
12	Algebraic stress models	2.2.

Summary of last lecture

Lecture 2 – Equations of fluid motion

- How can the fluid motion be described mathematically?
  - Navier-Stokes equations (momentum + continuity)
  - alternatively: vorticity equation
  - ⇒ energy, enstrophy equations
- What are the transformation properties of the conservation laws?
  - *Re* similarity, rotation/reflection, Galilean invariance
  - NO time reversal, Coriolis in rotating frame

LECTURE 3

Statistical description of turbulence

Questions to be answered in the present lecture

How do we compute and analyze a turbulent flow statistically?

Part I: What are the basic mathematical tools?

Part II: What are the averaged equations?

The need for a statistical description

Laminar flow

- velocity field $\mathbf{U}(\mathbf{x}, t)$ can be determined with accuracy

Turbulent flow

- velocity field $\mathbf{U}(\mathbf{x}, t)$ is random
  - perturbations in boundary/initial conditions are unavoidable
  - equations are extremely sensitive at high Reynolds

$\Rightarrow$ aim can only be a statistical description
  - a posteriori statistics $\longrightarrow$ data analysis
  - a priori statistics $\longrightarrow$ turbulence modelling

Random variables

Consider an experiment where a random variable u is measured

- probability of event $B \equiv \{u < v_b\}$

$$p = P(B) = P\{u < v_b\}\,, \qquad 0 \le p \le 1$$

- "CDF" – cumulative distribution function

$$F(V) \equiv P\{u < V\}$$

$$F(-\infty) = 0\,, \quad F(+\infty) = 1\,, \quad F(V_b) \ge F(V_a) \quad \text{if} \quad V_b > V_a$$

- "PDF" – probability density function

$$f(V) \equiv \frac{\mathrm{d}F(V)}{\mathrm{d}V}\,, \qquad f(V) \ge 0\,, \quad \int_{-\infty}^{+\infty} f(V)\mathrm{d}V = 1$$

Means and moments

- the mean of a random variable u

$$\langle u \rangle \equiv \int_{-\infty}^{+\infty} V\, f(V)\, \mathrm{d}V$$

- fluctuation of u

$$u' \equiv u - \langle u \rangle$$

- nth moment of u

$$\langle (u')^n \rangle \equiv \int_{-\infty}^{+\infty} (V - \langle u \rangle)^n\, f(V)\, \mathrm{d}V$$

- rules for averaging (a constant; Q, R random variables)

$$\langle\langle Q \rangle\rangle = \langle Q \rangle\,, \quad \langle a \rangle = a\,, \quad \langle Q + R \rangle = \langle Q \rangle + \langle R \rangle$$

Example probability distribution function

Normal (Gauss) distribution

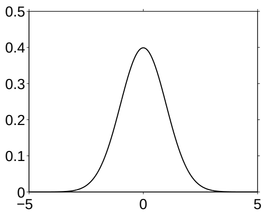
- with standard deviation σ, mean μ:

$$f(V) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(V-\mu)^2}{2\sigma^2}\right)$$

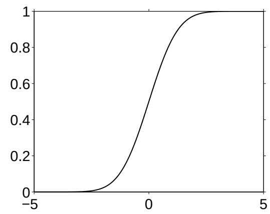
- standardized form, with $\hat{u} \equiv (u-\mu)/\sigma$:

$$\hat{f}(V) = \frac{1}{\sqrt{2\pi}} \exp\left(-V^2/2\right) , \quad \hat{F}(V) = \frac{1}{2}\left(1 + \text{erf}(V/\sqrt{2})\right)$$

$\hat{f}(V)$



$\hat{F}(V)$



Actual PDFs found in turbulent flow

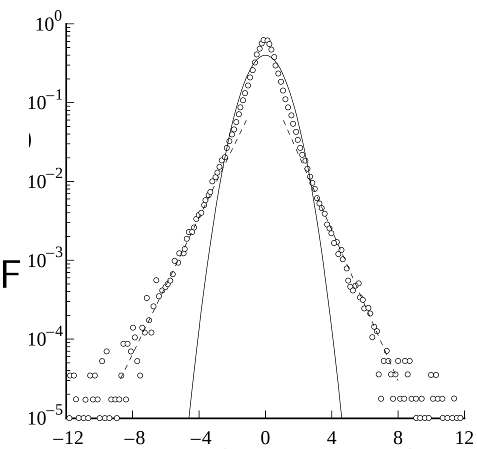
In general: turbulent quantities NOT Gaussian distributed!

Example:

- atmospheric boundary layer
- high *Re*
- axial velocity derivative

⇒ slowly decaying "tails"

PDF



$\partial_x u$ (normalized)

van Atta & Chen (1970)

Joint random variables

- multiple random variables are described jointly
- definitions analogous to single random variable
- cumulative distribution function of u_1, u_2:

$$F_{12}(V_1, V_2) \equiv P\{u_1 < V_1, u_2 < V_2\}$$

- "JPDF" – joint probability density function of u_1, u_2:

$$f_{12}(V_1, V_2) \equiv \frac{\partial^2}{\partial V_1 \partial V_2} F_{12}(V_1, V_2)$$

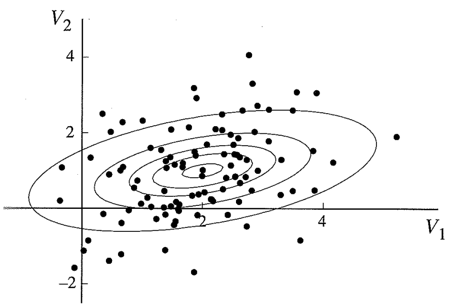
$$P\{V_{1a} \leq u_1 < V_{1b}, V_{2a} \leq u_2 < V_{2b}\} = \int_{V_{1a}}^{V_{1b}} \int_{V_{2a}}^{V_{2b}} f_{12}(V_1, V_2) \mathrm{d}V_1 \, \mathrm{d}V_2$$

JPDF & covariance

the example shows two random variables u_1, u_2 with:

- $\langle u_1 \rangle = 2$, $\langle u_2 \rangle = 1$
- $\langle u_1' u_1' \rangle = 1$, $\langle u_2' u_2' \rangle = 5/16$
- $\rho_{12} = 1/\sqrt{5}$

scatter plot – isocontours of JPDF



(from Pope, "Turbulence")

- covariance:
$$\langle u_1' u_2' \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (V_1 - \langle u_1 \rangle)(V_2 - \langle u_2 \rangle) f_{12}(V_1, V_2) \mathrm{d}V_1 \mathrm{d}V_2$$
- correlation coefficient: $\rho_{12} \equiv \langle u_1' u_2' \rangle / \sqrt{\langle u_1' u_1' \rangle \langle u_2' u_2' \rangle}$

Random processes

Describes a time-dependent random variable $u(t)$

- one-time CDF and PDF of random process:

$$F(V,t) \equiv P\{u(t) < V\}\,, \quad f(V;t) \equiv \frac{\mathrm{d}F(V,t)}{\mathrm{d}V}$$

$\longrightarrow$ no information on time correlation!

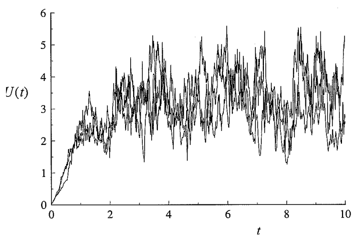
- define N-time joint CDF of $u(t)$:

$$F_N(V_1,t_1;V_2,t_2;\ldots) \equiv P\{u(t_1) < V_1; u(t_2) < V_2; \ldots; u(t_N) < V_N\}$$

- statistically stationary process:

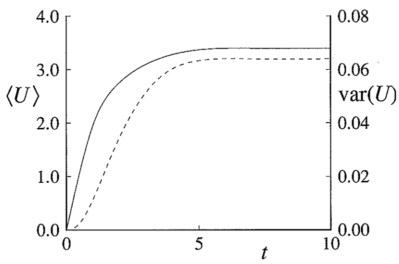
$$F_N(V_1,t_1+T;V_2,t_2+T;\ldots) = F_N(V_1,t_1;V_2,t_2;\ldots)$$

Statistically stationary random process



example:

Pope, "Turbulence"



- definition of a two-time autocorrelation function:

$$\rho(s) \equiv \langle u'(t)\,u'(t+s)\rangle / \langle u'(t)^2\rangle, \quad \rho(0)=1,\ |\rho(s)| \leq 1$$

$\Rightarrow$ correlation coefficient of process between t and $t+s$

- integral time scale: $\bar{\tau} = \displaystyle\int_0^\infty \rho(s)\,\mathrm{d}s$

Random fields

Time- and space-dependent vector field $\mathbf{u}(\mathbf{x}, t)$

- one-point, one-time JCDF, JPDF

$$F(\mathbf{V}, \mathbf{x}, t) \equiv P\{u_i(\mathbf{x}, t) < V_i, i = 1, 2, 3\}\,, \quad f(\mathbf{V}; \mathbf{x}, t) \equiv \frac{\partial^3 F(\mathbf{V}, \mathbf{x}, t)}{\partial V_1 \partial V_2 \partial V_3}$$

- mean value:

$$\langle \mathbf{u}(\mathbf{x}, t) \rangle = \int_{-\infty}^{\infty} \mathbf{V}\, f(\mathbf{V}, \mathbf{x}, t)\, \mathrm{d}V_1 \mathrm{d}V_2 \mathrm{d}V_3$$

- averaging commutes with differentiation:

$$\langle \partial_t u_i \rangle = \partial_t \langle u_i \rangle\,, \quad \langle \frac{\partial u_i}{\partial x_j} \rangle = \frac{\partial \langle u_i \rangle}{\partial x_j}$$

Statistical stationarity and homogeneity

Statistics invariant in time $\Rightarrow$ statistically stationary $\mathbf{u}(\mathbf{x}, t)$

Statistics invariant in space $\Rightarrow$ statistically homogeneous

- $\langle \mathbf{u} \rangle$ uniform in space

Statist. homogeneous $\mathbf{u}'(\mathbf{x}, t)$ $\Rightarrow$ homogeneous turbulence

- $\partial_j \langle u_i \rangle \neq 0$, but uniform in space
- homogeneity in 1, 2 or 3 dimensions

Isotropy: statistics invariant under rotation/reflection

Spatial statistics

- Two-point (one-time) correlation

$$R_{ij}(\mathbf{r},\mathbf{x},t) \equiv \langle u_i(\mathbf{x},t)\, u_j(\mathbf{x}+\mathbf{r},t)\rangle$$

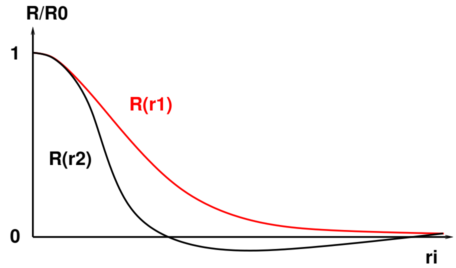
- integral length scale in direction $\mathbf{e}_k$:

$$L_{ij}^{(k)} = \frac{1}{R_{ij}(0,\mathbf{x},t)} \int_0^\infty R_{ij}(\mathbf{e}_k r, \mathbf{x}, t)\, \mathrm{d}r$$

$\rightarrow$ extension of largest scales

example:

- $L^{(1)} > L^{(2)}$



Wavenumber spectra in homogeneous turbulence

- homogeneous turbulence: Fourier analysis meaningful
- 3D spatial Fourier mode (wavevector $\boldsymbol{\kappa}$):

$$exp(I\boldsymbol{\kappa}\cdot\mathbf{x}) = \cos(\boldsymbol{\kappa}\cdot\mathbf{x}) + I\sin(\boldsymbol{\kappa}\cdot\mathbf{x})$$

- Fourier transform:

$$\hat{u}_i(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} u_i(\mathbf{x})\, e^{-I\boldsymbol{\kappa}\cdot\mathbf{x}}\mathrm{d}\mathbf{x}\,, \quad u_i(\mathbf{x}) = \int_{-\infty}^{\infty} \hat{u}_i(\boldsymbol{\kappa})\, e^{+I\boldsymbol{\kappa}\cdot\mathbf{x}}\mathrm{d}\boldsymbol{\kappa}$$

- homogeneity also implies: $R_{ij}(\mathbf{x},\mathbf{r},t) = R_{ij}(\mathbf{r},t)$

Wavenumber spectra in homogeneous turbulence (2)

- velocity spectrum tensor: transform of two-point correlation

$$\Phi_{ij}(\boldsymbol{\kappa}, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{-I\boldsymbol{\kappa}\cdot\mathbf{r}} R_{ij}(\mathbf{r}, t)\, \mathrm{d}\mathbf{r}$$

$$R_{ij}(\mathbf{r}, t) = \int_{-\infty}^{\infty} e^{+I\boldsymbol{\kappa}\cdot\mathbf{r}} \Phi_{ij}(\boldsymbol{\kappa}, t)\, \mathrm{d}\boldsymbol{\kappa}$$

- setting $\mathbf{r} = 0$: $R_{ij}(0, t) = \langle u_i' u_j' \rangle = \int_{-\infty}^{\infty} \Phi_{ij}(\boldsymbol{\kappa}, t)\, \mathrm{d}\boldsymbol{\kappa}$
- energy spectrum function:

$$E(\kappa, t) \equiv \frac{1}{2} \int_{-\infty}^{\infty} \Phi_{ii}(\boldsymbol{\kappa}, t)\, \delta(|\boldsymbol{\kappa}| - \kappa)\, \mathrm{d}\boldsymbol{\kappa}$$

- contribution to TKE: $\int_{-\infty}^{\infty} E(\kappa, t) \mathrm{d}\kappa = \frac{1}{2} R_{ii}(0, t) = \frac{1}{2} \langle u_i' u_i' \rangle$

Computing averages

- probabilistic definition

$$\langle u(t) \rangle \equiv \int_{-\infty}^{\infty} V\, f(V, t)\, \mathrm{d}V$$

- ensemble averaging (repeated experiments):

$$\langle u(t) \rangle_N \equiv \frac{1}{N} \sum_{n=1}^{N} u^{(n)}(t)$$

- time averaging (statist. stationary flow):

$$\langle u(t) \rangle_T \equiv \frac{1}{T} \int_t^{t+T} u(t')\, \mathrm{d}t'$$

- space averaging (homogeneous flow):

$$\langle u(t) \rangle_{\mathcal{L}} \equiv \frac{1}{\mathcal{L}^3} \int_{\mathcal{L}} u(\mathbf{x}, t)\, \mathrm{d}\mathbf{x}$$

Reynolds decomposition

- decompose velocity field into mean and fluctuation:

$$\mathbf{u}(\mathbf{x}, t) = \langle \mathbf{u}(\mathbf{x}, t) \rangle + \mathbf{u}'(\mathbf{x}, t)$$

- mean and fluctuation both satisfy continuity:

$$\nabla \cdot \langle \mathbf{u} \rangle = 0 \,, \quad \nabla \cdot \mathbf{u}' = 0$$

- averaged momentum equation:

$$\partial_t \langle u_i \rangle + (\langle u_i \rangle \langle u_j \rangle)_{,j} + \frac{1}{\rho} \langle p \rangle_{,i} = +\nu \langle u_i \rangle_{,jj} - \langle u_i' u_j' \rangle_{,j}$$

- appearance of Reynolds stress: $\langle u_i' u_j' \rangle_{,j}$

Characteristics of the Reynolds stresses

- Reynolds stresses act as additional or “apparent” stresses
- $\langle u_i' u_j' \rangle$ often much larger than viscous stresses
- closure problem: 4 equations – 10 unknowns !
- symmetric 2nd order tensor $\langle u_i' u_j' \rangle = \langle u_j' u_i' \rangle$
- turbulent kinetic energy (TKE): $k \equiv \frac{1}{2} \langle u_i' u_i' \rangle \geq 0$
- define anisotropy: $b_{ij} \equiv \dfrac{\langle u_i' u_j' \rangle}{2k} - \dfrac{\delta_{ij}}{3}$

  $\rightarrow \langle u_i' u_j' \rangle = 2k \left(\frac{1}{3} \delta_{ij} + b_{ij} \right)$

Closure by means of a turbulent viscosity

Turbulent viscosity hypothesis (Boussinesq 1877)

- $-\langle u_i' u_j' \rangle = \nu_T \left(\langle u_i \rangle_{,j} + \langle u_j \rangle_{,i} \right) - \frac{2}{3} \delta_{ij} \, k$

$\Rightarrow \partial_t \langle u_i \rangle + \left(\langle u_i \rangle \langle u_j \rangle \right)_{,j} + \frac{1}{\rho} \left(p_{eff} \right)_{,i} = \left[\nu_{eff} \left(\langle u_i \rangle_{,j} + \langle u_j \rangle_{,i} \right) \right]_{,j}$

$\nu_{eff} \equiv \nu + \nu_T \qquad p_{eff} = \langle p \rangle + \frac{2}{3} \rho k$

- with expression for ν_T & k, closed system $\rightarrow$ widely used
- BUT: often hypothesis not valid!

More details on RANS modelling in lectures 8-12

Summary

Main questions of the present lecture

- How do we compute and analyze a turbulent flow statistically?
- Part I: What are the basic mathematical tools?
  - random variables, processes, fields
  - two-time/two-point correlations
  - Fourier space analysis
- Part II: What are the averaged equations?
  - Reynolds decomposition and averaging
  - closure problem, turbulent viscosity

Problem

Consider the following Reynolds-stress tensors:

$$\langle u_i' u_j' \rangle = \begin{bmatrix} 0.6 & 0.05 & 0 \\ 0.05 & 0.5 & 0 \\ 0 & 0 & -0.1 \end{bmatrix} \qquad \langle u_i' u_j' \rangle = \begin{bmatrix} 0.4 & 0.5 & 0 \\ 0.5 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}$$

Why are these values impossible?

Outlook on next lecture: Free shear flows

How does a turbulent flow develop away from solid boundaries?

How can the equations be simplified for slow spatial evolution?

What is the evolution in the self-similar region?

What is the turbulence structure in a plane jet?

Further reading

- S. Pope, *Turbulent flows*, 2000
  $\longrightarrow$ chapter 3,4