Turbulenzmodelle in der Strömungsmechanik

Turbulent flows and their modelling

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Summary of last lecture

Lecture 1 – Introduction

- Why is fluid turbulence important?
  - ubiquity; greatly enhances transport (mass, momentum, heat)
- How do we define turbulence?
  - irregular, 3D, vortical, wide range of scales (Reynolds number)
- What are the principal difficulties for engineers?
  - unpredictable in detail, direct simulation too costly
  - LES, RANS approach more feasible

$\Rightarrow$ need for closure models

Equations of fluid motion
Transformation properties
Film
Conclusion

LECTURE 2

Equations of fluid motion

Questions to be answered in the present lecture

How can the fluid motion be described mathematically?

What are the transformation properties of the conservation laws?

Film "Turbulence" by R.W. Stewart

General assumptions

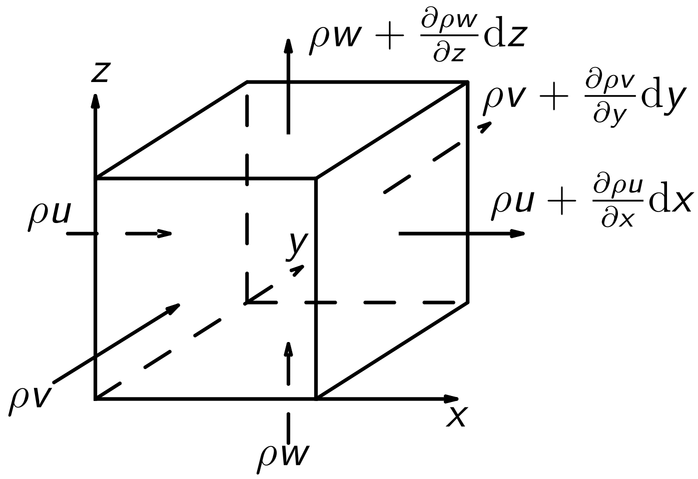
Set the following framework:

- continuum hypothesis ($Kn \equiv \lambda/\ell \ll 1$)
- consider Newtonian fluids
- restrict to incompressible fluids

Invoke the following principles:

- mass conservation
- Newton's second law

Mass conservation: continuity equation



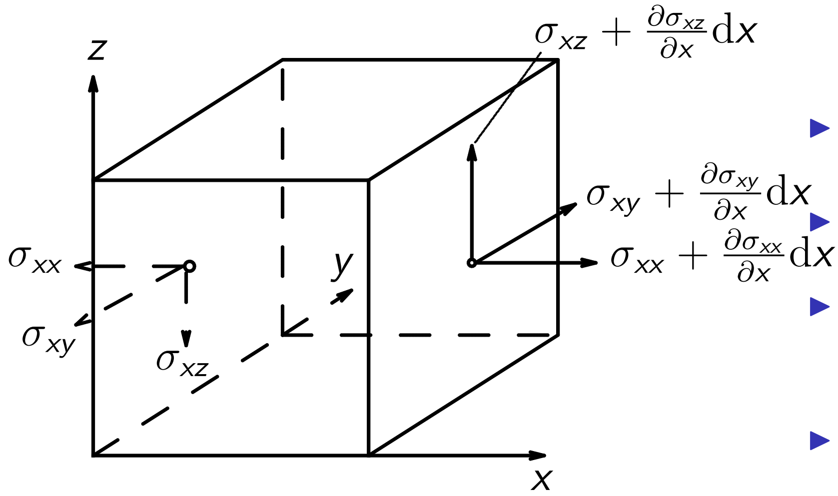
- time: t
- space: $\mathbf{x} = (x, y, z)$
- density: $\rho(\mathbf{x}, t)$
- velocity: $\mathbf{u}(\mathbf{x}, t) = (u, v, w)$

- $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho\, u_i)}{\partial x_i} = 0$ incompressibility: $\boxed{\dfrac{\partial u_i}{\partial x_i} = 0}$

Newton's 2nd law: momentum equation

Momentum balance

- $$\underbrace{\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}}_{\text{material acceleration}} = \underbrace{\rho\,\mathbf{f}}_{\text{body forces}} + \underbrace{\mathbf{P}}_{\text{surface forces}}$$



- $\dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}$
- $\mathbf{P} = -\nabla p + \nabla\cdot\boldsymbol{\tau}$
- Newtonian fluid: $\tau_{ij} = \mu(u_{i,j} + u_{j,i})$
- gravity: $\mathbf{f} = \mathbf{g}$

Momentum equation

Navier-Stokes equation for incompressible Newtonian fluid:

- $\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \mathbf{u} + \mathbf{f}$
- alternatively, in tensor notation:

  $$\partial_t u_i + u_j u_{i,j} + \frac{1}{\rho}\, p_{,i} = \nu\, u_{i,jj} + f_i$$
- constant density: gravity absorbed in modified pressure:

  $$\boxed{\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \mathbf{u}}$$

Equations of fluid motion
Transformation properties
Film
Conclusion
Continuity
Momentum
Kinetic energy
Vorticity

The role of pressure

Consider constant-density flows

- pressure $\neq$ thermodynamic variable
- pressure enforces divergence-free condition ($\Delta \equiv \nabla \cdot \mathbf{u}$):

  $$\frac{\mathrm{D}\Delta}{\mathrm{D}t} - \nu \nabla^2 \Delta = \underbrace{-\frac{1}{\rho} \nabla^2\, p - u_{j,i}\, u_{i,j}}_{\equiv R}$$

  for $\Delta = 0$, need $R = 0$. Therefore: $\nabla^2 p = -\rho(u_{j,i}\, u_{i,j})$
- non-local character of pressure:

  $$p(\mathbf{x}, t) = p^h(\mathbf{x}, t) + \frac{\rho}{4\pi} \int \frac{u_{j,i}\, u_{i,j}}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}\mathbf{x}'$$

$\Rightarrow$ perturbations affect the entire flow

The kinetic energy equation

Derivation

- definition of kinetic energy:
  $E_k \equiv \mathbf{u}\cdot\mathbf{u}/2 = u_i\, u_i/2$ (summation on i)
- transport equation: multiply momentum equation with $\mathbf{u}$

$$\frac{\mathrm{D}E_k}{\mathrm{D}t} = -\left(u_j\, p/\rho\right)_{,j} + \left(2\nu u_i S_{ij}\right)_{,j} - 2\nu S_{ij}S_{ij}$$

with: $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

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The kinetic energy equation

Significance of the terms

- $\dfrac{\partial E_k}{\partial t} = -\left(E_k u_j\right)_{,j} - \left(\dfrac{p}{\rho}u_j\right)_{,j} + \left(2\nu u_i S_{ij}\right)_{,j} - 2\nu S_{ij}S_{ij}$
- integration over an arbitrary volume V:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int E_k \mathrm{d}V = -\underbrace{\oint \left(E_k \mathbf{u}\cdot\mathbf{n}\right)\mathrm{d}S}_{\textit{convection}} - \underbrace{\oint \left(\frac{p}{\rho}\mathbf{u}\cdot\mathbf{n}\right)\mathrm{d}S}_{\textit{pressure work}} + \underbrace{\oint \left(\mathbf{u}\cdot\boldsymbol{\tau}\right)\cdot\mathbf{n}\,\mathrm{d}S}_{\textit{viscous work}} - \underbrace{\int \left(2\nu S_{ij}S_{ij}\right)\mathrm{d}V}_{\textit{dissipation}}$$

- rate of dissipation: $\varepsilon \equiv 2\nu S_{ij}S_{ij} \geq 0$

The vorticity equation

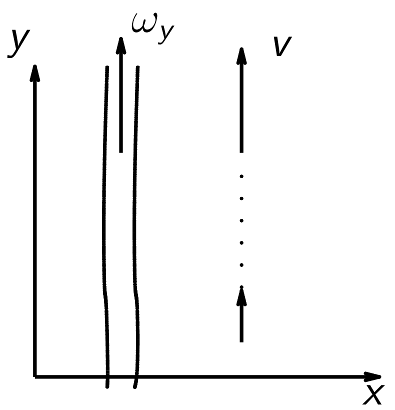
Definition of vorticity

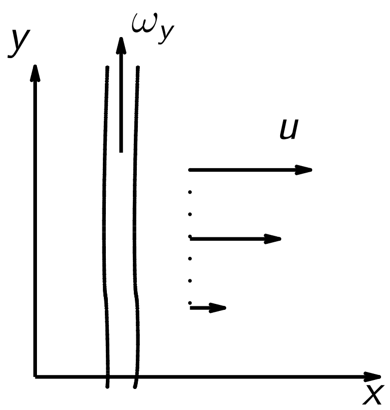
- $\boldsymbol{\omega} = \nabla \times \mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)^T$
- (twice) angular velocity of fluid element
- derive transport equation by applying curl to momentum equation:

  $\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla)\, \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\, \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}$
- pressure eliminated, additional stretching term

Vortex stretching and tilting

Contributions to the term $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$





- $\omega_y\, \partial_y(v) > 0$

$\longrightarrow$ tends to increase ω_y

- "vortex stretching"

- $\omega_y\, \partial_y(u) > 0$

$\longrightarrow$ tends to generate ω_x

- "vortex tilting"

The enstrophy equation

Enstrophy and vortex stretching

- definition of enstrophy: $\omega^2/2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega}/2$
- transport equation: multiply vorticity equation by $\boldsymbol{\omega}$

$$\frac{\mathrm{D}(\omega^2/2)}{\mathrm{D}t} = \boldsymbol{\omega} \cdot (\boldsymbol{\omega} \cdot \nabla)\, \mathbf{u} + \nu \nabla^2 (\omega^2/2) - \nu (\nabla \boldsymbol{\omega})^2$$

- in a turbulent flow:
  the stretching term tends to increase the enstrophy

Non-dimensional equations

- use length scale $\mathcal{L}$ and velocity scale $\mathcal{U}$
- define non-dimensional variables:
  $\hat{\mathbf{x}} = \mathbf{x}/\mathcal{L}$, $\hat{t} = t/\mathcal{T} = t\mathcal{U}/\mathcal{L}$
  $\hat{\mathbf{u}} = \mathbf{u}/\mathcal{U}$, $\hat{p} = p/(\rho\mathcal{U}^2)$
- substitute into Navier-Stokes equations:

$$\frac{\partial \hat{u}_i}{\partial \hat{x}_i} = 0$$

$$\frac{\partial \hat{u}_i}{\partial \hat{t}} + \hat{u}_j \frac{\partial \hat{u}_i}{\partial \hat{x}_j} = -\frac{\partial \hat{p}}{\partial \hat{x}_i} + \frac{1}{Re} \frac{\partial^2 \hat{u}_i}{\partial \hat{x}_j \partial \hat{x}_j}$$

- Reynolds number: $Re \equiv \dfrac{\mathcal{U}\mathcal{L}}{\nu}$

Transformation properties

Navier Stokes equations

- Reynolds number similarity
  (different $\mathcal{L}_b$, $\mathcal{U}_b$, ν_b only changes Re_b)
- time and space invariance
  (shift by $\mathbf{X}$ and T)
- rotated or reflected reference frame $\rightarrow$ invariant
- time reversal: NOT invariant
  (viscous term changes sign!)
- Galilean invariance (constantly moving frame)
  $\rightarrow$ Navier-Stokes are invariant (velocities are not)
- frame rotating in time $\rightarrow$ not invariant (Coriolis force)

Film material

"Turbulence"

- by R.W. Stewart, University of British Columbia
- 29 min
- US Natl. Commitee on Fluid Mechanics Films (1969)
- (play)

Summary

Main questions of the present lecture

- How can the fluid motion be described mathematically?
  - Navier-Stokes equations (momentum + continuity)
  - alternatively: vorticity equation
  - $\Rightarrow$ energy, enstrophy equations
- What are the transformation properties of the conservation laws?
  - *Re* similarity, rotation/reflection, Galilean invariance
  - NO time reversal, Coriolis in rotating frame

Outlook on next lecture: Statistical description

How do we compute and analyze a turbulent flow statistically?

Part I: What are the basic mathematical tools?

Part II: What are the averaged equations?

Further reading

- S. Pope, *Turbulent flows*, 2000
  → chapter 2
- U. Frisch, *Turbulence*, 1995
  → chapter 2