Turbulenzmodelle in der Strömungsmechanik

Turbulent flows and their modelling

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LECTURE 12

Algebraic stress models

Questions to be answered in the present lecture

How can the linear eddy viscosity assumption be avoided *without* the need for solving transport equations?

- algebraic stress models
- nonlinear eddy viscosity models

Intermediate between RSM and Boussinesq approximation

Reynolds stress transport models

- naturally incorporate transport effects
- describe stress production exactly

BUT: high computational cost (equations for 6 components)

Standard eddy-viscosity models (Boussinesq approximation)

(A) *local* relation between Reynolds stress and mean strain

(B) *linear* relation between Reynolds stress and mean strain

$\rightsquigarrow$ (A) is inevitable

$\Rightarrow$ (B) can be changed

$\Rightarrow$ non-linear Reynolds stress/mean strain relationships

Algebraic stress models

Reynolds stress equations for transport model:

$$\underbrace{\frac{\bar{\mathrm{D}}\langle u_i' u_j' \rangle}{\bar{\mathrm{D}} t} + \left(\mathcal{T}_{kij}\right)_{,k}}_{\equiv \mathcal{D}_{ij} \text{ (transport)}} = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \frac{2}{3}\tilde{\varepsilon}\delta_{ij}$$

Basic idea of algebraic stress models (ASM):

- approximating transport terms $\mathcal{D}_{ij}$ by *local* expressions

→ resulting model is free from derivatives:

6 *algebraic* equations relating $\langle u_i' u_j' \rangle$, k, $\tilde{\varepsilon}$, $\langle u_i \rangle_{,j}$

⇒ approach benefits from known models for pressure-strain $\mathcal{R}_{ij}$

Algebraic stress models – equilibrium assumption

Reynolds stress equations for transport model:

$$\underbrace{\frac{\bar{\mathrm{D}}\langle u_i' u_j' \rangle}{\bar{\mathrm{D}}t} + \left(\mathcal{T}_{kij}\right)_{,k}}_{\equiv \mathcal{D}_{ij}\,\text{(transport)}} = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \frac{2}{3}\tilde{\varepsilon}\delta_{ij}$$

Simplest local equilibrium assumption:

- ▶ neglect the transport term altogether: $\mathcal{D}_{ij} = 0$
- ⇒ implies for the turbulent energy: $\frac{1}{2}\mathcal{D}_{ll} = \mathcal{P} - \tilde{\varepsilon} = 0$
- ⇝ problem: equality $\mathcal{P} = \tilde{\varepsilon}$ not verified in general!

Algebraic stress models – weak equilibrium assumption

Reynolds stress equations for transport model:

$$\underbrace{\frac{\bar{\mathrm{D}}\langle u_i' u_j' \rangle}{\bar{\mathrm{D}}t} + \left(\mathcal{T}_{kij}\right)_{,k}}_{\equiv \mathcal{D}_{ij}\ \text{(transport)}} = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \frac{2}{3}\tilde{\varepsilon}\delta_{ij}$$

Weak equilibrium assumption (Rodi 1972)

- rewriting Reynolds stress in terms of anisotropy and TKE:
  $\langle u_i' u_j' \rangle = 2k\, b_{ij} + \frac{2}{3} k\, \delta_{ij}$
- neglecting transport of anisotropy:
  $$\frac{\bar{\mathrm{D}}\langle u_i' u_j' \rangle}{\bar{\mathrm{D}}t} = \frac{\langle u_i' u_j' \rangle}{k}\frac{\bar{\mathrm{D}}k}{\bar{\mathrm{D}}t} + k\frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t}\left(\frac{\langle u_i' u_j' \rangle}{k}\right) \approx \frac{\langle u_i' u_j' \rangle}{k}\frac{\bar{\mathrm{D}}k}{\bar{\mathrm{D}}t}$$
- applying the approximation to the entire transport term:
  $$\mathcal{D}_{ij} \approx \frac{\langle u_i' u_j' \rangle}{k} \cdot (\text{transport of } k) = \frac{\langle u_i' u_j' \rangle}{k}\frac{1}{2}\mathcal{D}_{ll} = \frac{\langle u_i' u_j' \rangle}{k}\left(\mathcal{P} - \tilde{\varepsilon}\right)$$

⇒ final model: $\frac{\langle u_i' u_j' \rangle}{k}\left(\mathcal{P} - \tilde{\varepsilon}\right) = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \frac{2}{3}\tilde{\varepsilon}\delta_{ij}$

ASM predictions for homogeneous shear flow

LRR-IP pressure-strain model

$$\mathcal{R}_{ij} = -C_R 2\tilde{\varepsilon} b_{ij} - C_2(\mathcal{P}_{ij} - \tfrac{2}{3}\mathcal{P}\delta_{ij})$$

- corresponding ASM:

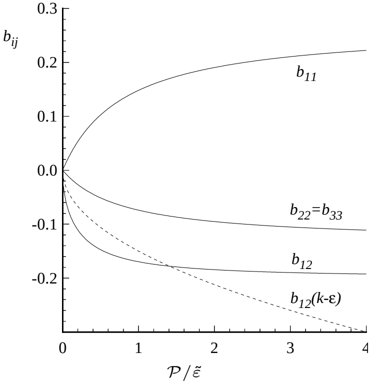
$$b_{ij} = \frac{\frac{1}{2}(1-C_2)}{C_R - 1 + \mathcal{P}/\tilde{\varepsilon}} \cdot \frac{\mathcal{P}_{ij} - \frac{2}{3}\delta_{ij}\mathcal{P}}{\tilde{\varepsilon}}$$

- in homogeneos shear flow:

b_{ij} has finite limit for $\frac{\mathcal{P}}{\tilde{\varepsilon}} \to \infty$

$$b_{11} \to \frac{4}{15}$$
$$b_{22} \to -\frac{2}{15}$$
$$b_{12} \to -\frac{1}{5}$$

$\Rightarrow$ stress remains realizable



——, ASM predictions; – – – –, k-ε model

(from Pope "Turbulent Flows", 2000)

Stress/mean strain relation implied by ASM

LRR-IP pressure-strain model

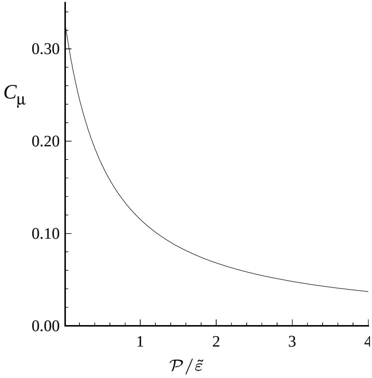
- define:

  $$-\langle u'v' \rangle = C_\mu \frac{k^2}{\tilde{\varepsilon}} \langle u \rangle_{,y}$$

  with unknown function C_μ
- substituting ASM:

  $$C_\mu = \frac{\frac{2}{3}(1-C_2)(C_R-1+C_2\mathcal{P}/\tilde{\varepsilon})}{(C_R-1+\mathcal{P}/\tilde{\varepsilon})^2}$$

$\Rightarrow$ C_μ decreases with $\mathcal{P}/\tilde{\varepsilon}$



——, ASM predictions

(from Pope "Turbulent Flows", 2000)

Assessing the ASM approach

Achievements of algebraic stress models

- partial differential equations reduced to *algebraic equations*
- physics of pressure-strain model is carried over

Problems of the ASM approach

- implicit system of equations
- dependence is in general non-linear
- system can have multiple solutions
- numerical stiffness

Explicit ASM or non-linear eddy viscosity models

Explicit ASM (EASM)

- *explicit* expressions for the stress components are numerically desirable
- there are two routes (viewpoints) to achieve this:

1. construct an *implicit* ASM (as above): $b_{ij} = f_i(b_{ij}, \frac{k}{\tilde{\varepsilon}} \langle u_i \rangle_{,j})$
   then derive equivalent *explicit* form analytically
   $\Rightarrow \qquad b_{ij} = f_e(\frac{k}{\tilde{\varepsilon}} \langle u_i \rangle_{,j})$
2. construct an *explicit* expression for the Reynolds stresses:
   $\Rightarrow \qquad b_{ij} = f_{e'}(\frac{k}{\tilde{\varepsilon}} \langle u_i \rangle_{,j})$

$\Rightarrow$ both approaches have been realized

$\Rightarrow$ results also known as "non-linear eddy viscosity models"

Deriving explicit algebraic stress models

- ansatz: $$b_{ij} = \mathcal{B}_{ij}\left(\widehat{\mathbf{S}}, \widehat{\boldsymbol{\Omega}}\right)$$

  where normalized mean rate of strain/rotation are defined:

  $\widehat{S}_{ij} \equiv \frac{k}{2\tilde{\varepsilon}}\left(\langle u_i\rangle_{,j} + \langle u_j\rangle_{,i}\right)$, $\widehat{\Omega}_{ij} \equiv \frac{k}{2\tilde{\varepsilon}}\left(\langle u_i\rangle_{,j} - \langle u_j\rangle_{,i}\right)$

- most general consistent expression (Pope 1975):

$$\mathcal{B}_{ij}\left(\widehat{\mathbf{S}}, \widehat{\boldsymbol{\Omega}}\right) = \sum_{n=1}^{10} G^{(n)} \widehat{\mathcal{T}}_{ij}^{(n)}$$

- with independent, symmetric, deviatoric functions:

$\widehat{\mathcal{T}}_{ij}^{(1)} = \widehat{\mathbf{S}}$ $\quad\widehat{\mathcal{T}}_{ij}^{(2)} = \widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}} - \widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}$ $\quad\widehat{\mathcal{T}}_{ij}^{(3)} = \widehat{\mathbf{S}}^2 - \frac{1}{3}\mathrm{trace}(\widehat{\mathbf{S}}^2)\mathbf{I}$

$\widehat{\mathcal{T}}_{ij}^{(4)} = \widehat{\boldsymbol{\Omega}}^2 - \frac{1}{3}\mathrm{trace}(\widehat{\boldsymbol{\Omega}}^2)\mathbf{I}$ $\quad\widehat{\mathcal{T}}_{ij}^{(5)} = \widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}^2 - \widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}$ $\quad\widehat{\mathcal{T}}_{ij}^{(6)} = \widehat{\boldsymbol{\Omega}}^2\widehat{\mathbf{S}} + \widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}}^2 - \frac{2}{3}\mathrm{trace}(\widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}}^2)\mathbf{I}$

$\widehat{\mathcal{T}}_{ij}^{(7)} = \widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}}^2 - \widehat{\boldsymbol{\Omega}}^2\widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}}$ $\quad\widehat{\mathcal{T}}_{ij}^{(8)} = \widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}^2 - \widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}$ $\quad\widehat{\mathcal{T}}_{ij}^{(9)} = \widehat{\boldsymbol{\Omega}}^2\widehat{\mathbf{S}}^2 + \widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}^2 - \frac{2}{3}\mathrm{trace}(\widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}^2)\mathbf{I}$

$\widehat{\mathcal{T}}_{ij}^{(10)} = \widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}^2 - \widehat{\boldsymbol{\Omega}}^2\widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}$

  and undetermined scalar coefficients $G^{(n)}$

Examples of EASM

Linear case – Boussinesq hypothesis

- $G^{(1)} = -C_\mu$; $G^{(n)} = 0$ for $n \geq 2 \quad \rightarrow \quad b_{ij} = -C_\mu \widehat{S}_{ij}$

Statistically two-dimensional flow (Pope, 1975)

- sum contains only three terms ($G^{(n)} = 0$ for $n \geq 4$)

General three-dimensional flow

- all 10 terms are non-zero

1. ASM approach:
   Gatski & Speziale (1993), based on linear pressure-strain
2. direct approach:
   Shih, Zhu & Lumley (1995), based on realizability

Performance of EASM in mixing layer flow

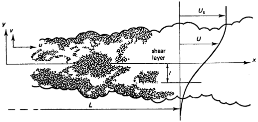
Self-similar mixing layer

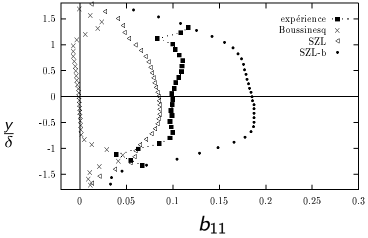
- spreading rate $d\delta/dx$:

exp.	EASM (SZL)	k-ε
0.019	0.014	0.016

$\rightsquigarrow$ EASM by Shih et al. not well calibrated for free shear flows

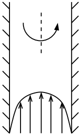
- anisotropy well predicted (normal stresses)





(experiment of Bell & Mehta, 1990)

Performance of EASM for rotating channel flow

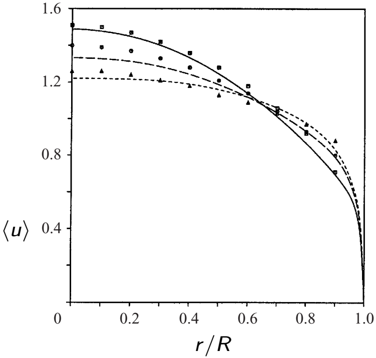


Rotation in *axial* direction

- rotation has stabilizing effect (production term)

$\Rightarrow$ EASM predictions are reasonable

$\rightsquigarrow$ linear eddy-viscosity fails

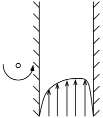


lines: EASM of Wallin & Johansson (2000)

symbols: experiment of Imao et al. (1996)

▲ no rotation; ● medium; ■ strong

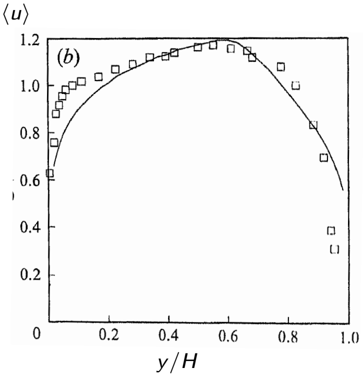
Performance of EASM for rotating channel flow (2)



Rotation in *spanwise* direction

- rotation causes non-symmetric profiles

$\Rightarrow$ EASM predictions are comparable to full transport model



——, EASM of Gatski & Speziale (1993)

☐ experiment of Johnston et al. (1972)

Summary of today's lecture

How can the linear eddy viscosity assumption be avoided *without* the need for solving transport equations?

- transport terms eliminated by weak equilibrium assumption

1. algebraic stress models (ASM)
   - inherit properties of pressure-strain model
   - $\rightsquigarrow$ often numerical difficulties
2. nonlinear eddy viscosity models (EASM)
   - provide general explicit expressions for Reynolds stresses
   - $\Rightarrow$ allow for prediction of complex straining fields

Summary (2): A hierarchy of RANS models

(a) elliptic relaxation RSM

(b) standard RSM

(c) ASM (with k-ε equations)

(d) nonlinear eddy-viscosity model (with k-ε)

(e) standard (isotropic eddy-viscosity) k-ε model

(f) one-equation k-model (with ℓ_m)

(g) mixing-length model

Which assumption is added when stepping from (a) to (g)?

Problem to be solved:

Consider a simple homogeneous shear flow with $\langle u_i \rangle_{,j} = \delta_{i1}\delta_{j2}S$*, where* S *is constant. Write the turbulent shear-stress anisotropy* b_{12} *given by the* k*-*ε *model as a function of the ratio between turbulent kinetic energy production and dissipation,* $\mathcal{P}/\tilde{\varepsilon}$*. What is the limiting value of* $\mathcal{P}/\tilde{\varepsilon}$ *above which the Reynolds stress tensor becomes non-realizable?*

Further reading

- S. Pope, *Turbulent flows*, 2000
  $\longrightarrow$ chapter 11
- D.C. Wilcox, *Turbulence modeling for CFD*, 2006
  $\longrightarrow$ chapter 6