Turbulenzmodelle in der Strömungsmechanik

Turbulent flows and their modelling

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LECTURE 10

Reynolds-stress transport models

Questions to be answered in the present lecture

How can the equations be closed at the second-moment level?

- why resort to Reynolds-stress models?
- how to derive the $\langle u_i' u_j' \rangle$ transport equation?
- how to model the principal unknown terms?

How do Reynolds-stress models perform?

Why use Reynolds-stress transport models?

Fundamental deficiency of turbulent viscosity models:

- Reynolds stress is assumed *local* function of mean strain-rate

$\longrightarrow$ transport/history effects are neglected
(e.g. failure in relaxation from mean strain – cf. lecture 8)

Attractive features of Reynolds-stress transport models:

- avoid any turbulent viscosity hypothesis
- transport & production terms are in *closed* form

$\longrightarrow$ transport effects "built-in"

$\longrightarrow$ stress production "exact" even in complex straining fields

Deriving the transport equation for the Reynolds stress

Steps in deriving the exact equation from Navier-Stokes

note that $\partial_t \langle u_i' u_j' \rangle = \langle u_j' \partial_t u_i' + u_i' \partial_t u_j' \rangle$

1. write transport equation for fluctuating velocity $\mathbf{u}'$
2. multiply ith-component with u_j'
3. multiply jth-component with u_i'
4. add results from 2. and 3.
5. take average of result from 4.

The exact transport equation for the Reynolds stress

$$\frac{\bar{D}\langle u_i' u_j'\rangle}{\bar{D}t} + \left[\underbrace{\langle u_i' u_j' u_k'\rangle + \frac{1}{\rho}\langle p' u_j'\rangle\delta_{ik} + \frac{1}{\rho}\langle p' u_i'\rangle\delta_{jk} - \nu\langle u_i' u_j'\rangle_{,k}}_{\text{turbulent transport } \mathcal{T}_{kij}}\right]_{,k} =$$

$$\underbrace{-\langle u_i\rangle_{,k}\langle u_k' u_j'\rangle - \langle u_j\rangle_{,k}\langle u_k' u_i'\rangle}_{\text{production } \mathcal{P}_{ij}} + \underbrace{\frac{1}{\rho}\left(\langle p' u_{j,i}'\rangle + \langle p' u_{i,j}'\rangle\right)}_{\text{pressure-strain } \mathcal{R}_{ij}} - \underbrace{2\nu\langle u_{i,k}' u_{j,k}'\rangle}_{\text{dissipation tensor } \varepsilon_{ij}}$$

- pressure–rate-of-strain $\mathcal{R}_{ij}$ and dissipation ε_{ij} are *unclosed*
- first three terms of turbulent transport $\mathcal{T}_{kij}$ are *unclosed*
- half the trace of this equation yields TKE equation
- pressure–rate-of-strain is absent in TKE equation

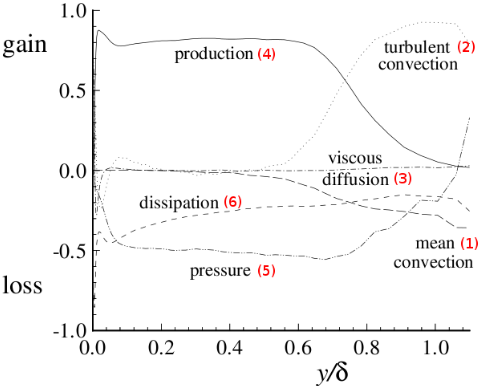
Importance of the terms in a boundary layer flow

$$0 = \underset{(1)}{-\frac{\bar{D}\langle u_i' u_j'\rangle}{\bar{D}t}} \quad \underset{(2)}{-\langle u_i' u_j' u_k'\rangle_{,k}} \quad \underset{(3)}{+\nu\langle u_i' u_j'\rangle_{,kk}} \quad \underset{(4)}{+\mathcal{P}_{ij}} \quad \underset{(5)}{+\Pi_{ij}} \quad \underset{(6)}{-\varepsilon_{ij}}$$

Budget of streamwise normal stress $\langle u'u'\rangle$

- principal production: $\mathcal{P}_{11} \approx -2\langle u'v'\rangle\langle u\rangle_{,y}$
- mainly balanced by: dissipation & pressure–rate-of-strain

note: $\Pi_{ij} \equiv \mathcal{R}_{ij} - \frac{1}{\rho}\left(\langle p' u_j'\rangle\delta_{ik} + \langle p' u_i'\rangle\delta_{jk}\right)_{,k}$



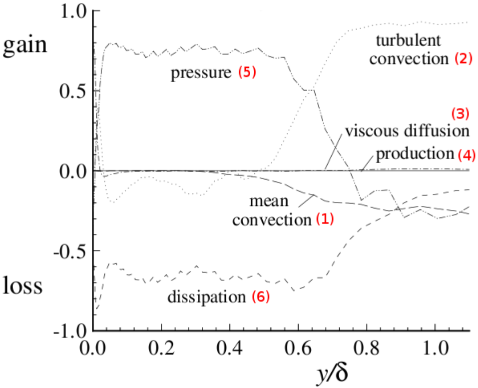
(DNS Spalart 1988, $Re_\theta = 1410$)

Importance of the terms in a boundary layer flow

$$0 = \underbrace{-\frac{\bar{D}\langle u_i' u_j'\rangle}{\bar{D}t}}_{(1)} \underbrace{-\langle u_i' u_j' u_k'\rangle_{,k}}_{(2)} \underbrace{+\nu\langle u_i' u_j'\rangle_{,kk}}_{(3)} \underbrace{+\mathcal{P}_{ij}}_{(4)} \underbrace{+\Pi_{ij}}_{(5)} \underbrace{-\varepsilon_{ij}}_{(6)}$$

Budget of wall-normal stress $\langle v'v'\rangle$

- no production: $\mathcal{P}_{22} \approx 0$
- gain from pressure–rate-of-strain
- approximately balanced by dissipation



(DNS Spalart 1988, $Re_\theta = 1410$)

Nature of the pressure–rate-of-strain correlations

Observation from flow data:

- pressure terms are of *significant* magnitude
- pressure–rate-of-strain correlation has *redistributive* character
- due to incompressibility, term has zero trace:

$$\mathcal{R}_{ij} \equiv \frac{1}{\rho}\left(\langle p' u'_{j,i}\rangle + \langle p' u'_{i,j}\rangle\right) \quad \rightarrow \quad \mathcal{R}_{ii} = 0$$

$\Rightarrow$ no contribution to turbulent kinetic energy

Pressure–rate-of-strain is main challenge for modelling!

Dissipation tensor components in boundary layer flow

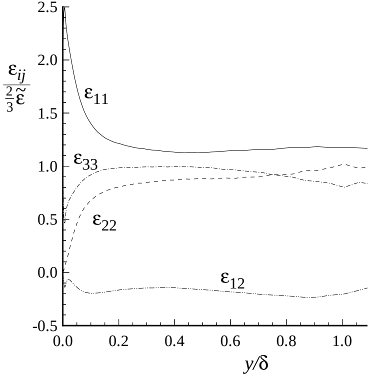
Observations:

- for high Reynolds number:

  dissipation tensor is approximately isotropic
- low Reynolds in DNS:

  $\longrightarrow$ some residual isotropy

but: significant anisotropy near wall (cf. lecture 11)



(DNS Spalart 1988, $Re_\theta = 1410$)

$\Rightarrow$ dissipation tensor often modelled as isotropic: $\boxed{\varepsilon_{ij} = \frac{2}{3}\tilde{\varepsilon}\,\delta_{ij}}$

Unclosed terms in the Reynolds-stress transport equation

Reynolds-stress equation, assuming isotropic dissipation:

$$\frac{\bar{\mathrm{D}}\langle u_i' u_j'\rangle}{\bar{\mathrm{D}}t} + \left[\underbrace{\langle u_i' u_j' u_k'\rangle}_{\mathcal{T}_{kij}^{(u)}} + \underbrace{\frac{1}{\rho}\langle p' u_j'\rangle \delta_{ik} + \frac{1}{\rho}\langle p' u_i'\rangle \delta_{jk}}_{\mathcal{T}_{kij}^{(p)}} + \nu \langle u_i' u_j'\rangle_{,k} \right]_{,k} = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \tfrac{2}{3}\tilde{\varepsilon}\delta_{ij}$$

Models need to be prescribed for the following terms:

- triple correlation $\mathcal{T}_{kij}^{(u)}$ and pressure transport $\mathcal{T}_{kij}^{(p)}$
- the scalar (pseudo) dissipation rate $\tilde{\varepsilon} \quad \rightarrow \quad$ similar k-ε model
- the pressure–rate-of-strain correlation $\mathcal{R}_{ij}$

The Poisson equation for pressure fluctuations

Fluctuating pressure equation (instantan. pressure: lecture 2)

$$\boxed{\frac{1}{\rho}\nabla^2 p' = -2\langle u_i\rangle_{,j}\, u'_{j,i} - \left(u'_i u'_j - \langle u'_i u'_j\rangle\right)_{,ij}}$$

- pressure can be decomposed into 3 contributions $p' = p^{(h)} + p^{(r)} + p^{(s)}$
- *homogeneous* pressure $p^{(h)}$: $\nabla^2 p^{(h)} = 0$
- *rapid* pressure $p^{(r)}$: $\nabla^2 p^{(r)} = -2\rho\langle u_i\rangle_{,j}\, u'_{j,i}$
- *slow* pressure $p^{(s)}$: $\nabla^2 p^{(s)} = -\rho\left(u'_i u'_j - \langle u'_i u'_j\rangle\right)_{,ij}$

$\Rightarrow$ 3 different contributions to pressure-strain correlation

Contributions to pressure-strain correlation

Homogeneous pressure $\longrightarrow$ $\mathcal{R}_{ij}^{(h)} \equiv \langle p^{(h)}(u'_{i,j} + u'_{j,i})\rangle/\rho$

- influenced by boundary conditions only
- $\mathcal{R}_{ij}^{(h)}$ vanishes in homogeneous turbulence
- contribution important near walls (lecture 11)

Rapid pressure

- reacts instantly to mean velocity gradients
- dominant contribution for large strain rate $\mathcal{S}k/\varepsilon$

Slow pressure

- determined by self-interaction of turbulent field
- principal mechanism for return to isotropy without strain

Modelling the slow part of pressure-strain

Homogeneous turbulence *without* mean velocity gradients:

$$\boxed{\partial_t \langle u_i' u_j' \rangle = \mathcal{R}_{ij} - \frac{2}{3}\tilde{\varepsilon}\delta_{ij}}$$

- no mean velocity gradients $\rightarrow \mathcal{R}_{ij} = \mathcal{R}_{ij}^{(s)}$
- modelling ansatz: $\mathcal{R}_{ij}^{(s)} = \tilde{\varepsilon}\,\mathcal{F}_{ij}\left(b_{ij}\right)$
- the most general tensor function is:

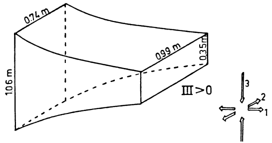
$$\mathcal{F}_{ij}^{(s)} = C_1 b_{ij} + C_2 \left(b_{ij}^2 - \frac{1}{3} b_{kk}^2\,\delta_{ij} \right)$$

- C_1, C_2 are scalar functions
- Rotta's linear model: $\boxed{C_1 = -2C_R, \quad C_2 = 0}$

$\Rightarrow$ linear return-to-isotropy: $\mathrm{d}_t b_{ij} = -(C_R - 1)\,\frac{\tilde{\varepsilon}}{k}\,b_{ij}$

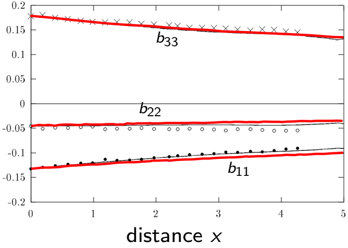
Slow pressure-strain models: comparison with experiment

Return-to-isotropy *after* distorting duct



- mean strain is imposed in a distorting duct
- then: in straight section, turbulence relaxes to isotropy
- simple <u>linear model works</u> in this case

evolution of b_{ij} in straight duct section



($\times$, $\circ$, $\bullet$ experiment LePenven et al. 1985)

(—— Rotta model, $C_R = 1.5$)

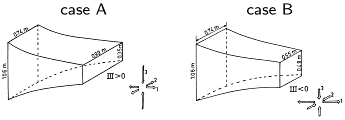
Return-to-isotropy: depends on complete state of **b**

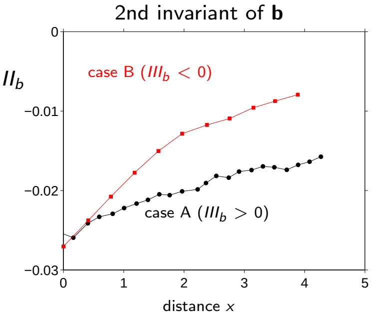
- process can be described by two invariants of tensor **b**:
  $II_b = -\frac{1}{2} b_{ij} b_{ji}$
  $III_b = \frac{1}{3} b_{ij} b_{jk} b_{ki}$
- isotropy: $II_b = III_b = 0$
- experiments: return rate depends on invariant III_b !
- linear model:
  $\mathrm{d}_t II_b = -2(C_R - 1)\frac{\tilde{\varepsilon}}{k} II_b$

$\rightsquigarrow$ model should be non-linear

$\Rightarrow$ Shih/Lumley 1985: $C_1(II, III)$

$\Rightarrow$ Speziale et al 1991: $C_2 \neq 0$





Modelling the fast part of pressure-strain

Homogeneous turbulence *with* mean velocity gradients:

$$\boxed{\partial_t \langle u_i' u_j' \rangle = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \frac{2}{3}\tilde{\varepsilon}\delta_{ij}}$$

- here: $\mathcal{P}_{ij} \neq 0$ and $\mathcal{R}_{ij} = \mathcal{R}_{ij}^{(r)} + \mathcal{R}_{ij}^{(s)}$
- exact expression: $\mathcal{R}_{ij}^{(r)} = 2\langle u_l \rangle_{,k} \left(\mathcal{M}_{kjil} + \mathcal{M}_{ikjl}\right)$
  where $\mathcal{M}_{ijkl}$ is an integral of two-point correlations
- in single-point closures: $\mathcal{M}_{iljk}$ modelled as function of $(\mathbf{b}, k)$
- symmetry, tensorial considerations & realizability constraints

$\longrightarrow$ lead to functional form:

$$\mathcal{R}_{ij}^{(r)} = k \sum_{n=3}^{8} f^{(n)} \mathcal{T}_{ij}^{(n)}$$

Some common rapid pressure-strain models:

$$\boxed{\mathcal{R}_{ij}^{(r)} = k \sum_{n=3}^{8} f^{(n)}\, \mathcal{T}_{ij}^{(n)}}$$

n	$\mathcal{T}_{ij}^{(n)}$	$\underline{f^{(n)}}$:	LRR (Launder et al. 1975)	SSG (Speziale et al. 1991)	SL (Shih & Lumley 1985)
3	$\bar{S}_{ij}$		$\frac{4}{5}$	$\frac{4}{5} + 1.3\sqrt{II_b}$	$\frac{4}{5}$
4	$\bar{S}_{ik} b_{kj} + b_{ik}\bar{S}_{kj} - \frac{2}{3}\bar{S}_{kl} b_{lk}\delta_{ij}$		$\frac{6}{11}(2+3C_2)$	$\frac{5}{4}$	$12C_2$
5	$\bar{\Omega}_{ik} b_{kj} - b_{ik}\bar{\Omega}_{kj}$		$\frac{2}{11}(10-7C_2)$	$\frac{2}{5}$	$\frac{4}{3}(2-7C_2)$
6	$\bar{S}_{ik} b_{kj}^2 + b_{ik}^2\bar{S}_{kj} - \frac{2}{3}\bar{S}_{kl} b_{lk}^2\delta_{ij}$		0	0	$\frac{4}{5}$
7	$\bar{\Omega}_{ik} b_{kj}^2 - b_{ik}^2\bar{\Omega}_{kj}$		0	0	$\frac{4}{5}$
8	$b_{ik}\bar{S}_{kl} b_{lj} - \frac{1}{3}\bar{S}_{kl} b_{lk}^2\delta_{ij}$		0	0	$-\frac{8}{5}$
			$C_2 = 0.4$		$C_2 = \frac{1}{10}(1+\frac{4}{5}g(II_b, III_b))$
	satisfies realizability:		no	yes	yes

where: $\bar{S}_{ij} \equiv \frac{1}{2}\left(\langle u_i\rangle_{,j} + \langle u_j\rangle_{,i}\right)$, $\bar{\Omega}_{ij} \equiv \frac{1}{2}\left(\langle u_i\rangle_{,j} - \langle u_j\rangle_{,i}\right)$

Results for pressure-strain models in homogeneous shear

Comparison with experimental measurements

- note: this test involves rapid *and* slow parts $\mathcal{R}_{ij}^{(r)} + \mathcal{R}_{ij}^{(s)}$
- equilibrium results:

	LRR	SSG	SL	experiment Tavoularis & Karnik (1989)
b_{11}	0.155	0.219	0.135	0.18
b_{22}	-0.122	-0.146	-0.136	-0.11
b_{33}	0.033	0.073	-0.001	0.07
b_{12}	-0.188	-0.164	-0.108	-0.16

- LRR and SSG models provide reasonable values
- Shih/Lumley model yields too weak tangential component b_{12}

Reynolds-stress transport in non-homogeneous flow

$$\frac{\bar{\mathrm{D}}\langle u_i' u_j'\rangle}{\bar{\mathrm{D}}t} + \Bigg[\underbrace{\langle u_i' u_j' u_k'\rangle}_{\mathcal{T}_{kij}^{(u)}} + \underbrace{\frac{1}{\rho}\langle p' u_j'\rangle\delta_{ik} + \frac{1}{\rho}\langle p' u_i'\rangle\delta_{jk}}_{\mathcal{T}_{kij}^{(p)}} + \nu\langle u_i' u_j'\rangle_{,k}\Bigg]_{,k} = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \tfrac{2}{3}\tilde{\varepsilon}\delta_{ij}$$

Additional modeling for non-homogeneous flows:

- use local pressure–strain model $\mathcal{R}_{ij}$ as in homogeneous flow (except for wall corrections – cf. lecture 11)
- models for triple correlation $\mathcal{T}_{kij}^{(u)}$ and pressure transport $\mathcal{T}_{kij}^{(p)}$
- equation for the dissipation rate, involving transport $\tilde{\varepsilon}$

Modeling combined turbulent transport

Gradient diffusion models $\mathcal{T}_{kij}^{(t)} = \mathcal{T}_{kij}^{(u)} + \mathcal{T}_{kij}^{(p)}$

- isotropic eddy diffusivity: $$\mathcal{T}_{kij}^{(t)} = -\underbrace{C_s \frac{k^2}{\tilde{\varepsilon}}}_{\nu_T} \frac{\partial \langle u_i' u_j' \rangle}{\partial x_k}$$

  with constant $C_s = 0.09$ as in k-ε model
- eddy diffusivity tensor model: $$\mathcal{T}_{kij}^{(t)} = -\underbrace{C_s \frac{k}{\tilde{\varepsilon}} \langle u_k' u_l' \rangle}_{(\nu_T)_{kl}} \frac{\partial \langle u_i' u_j' \rangle}{\partial x_l}$$

  with constant $C_s = 0.22$
- more general models, symmetric w.r.t. indices i, j, k (Mellor & Herring, 1973; Hanjalic & Launder, 1972)
- models based on transport equation for $\langle u_i' u_j' u_k' \rangle$

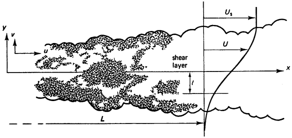
Experimental data for triple correlation

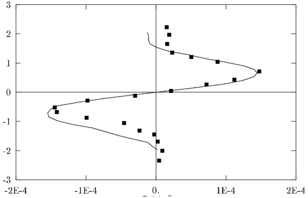
Self-similar mixing layer

- triple term dominates transport
- compare $\mathcal{T}^{(u)}_{kij}$ with isotropic eddy diffusivity model:

$$\langle u'v'v'\rangle = -C_s \frac{k^2}{\tilde{\varepsilon}} \langle u'v'\rangle_{,y}$$

$\Rightarrow$ reasonable predictions





$\frac{y}{r_{1/2}}$

$\langle u'v'v'\rangle / U_0^3$

■ measurement Bell & Mehta (1990)

—— eddy diffusivity model

Dissipation rate equation

Similar model equation as in k-ε model:

$$\frac{\bar{\mathrm{D}}\tilde{\varepsilon}}{\bar{\mathrm{D}}t} = \left(\left(\nu \cdot \delta_{lk} + \frac{(\nu_T)_{kl}}{\sigma_\varepsilon}\right)\tilde{\varepsilon}_{,k}\right)_{,l} + C_{\varepsilon_1}\frac{\mathcal{P}\tilde{\varepsilon}}{k} - C_{\varepsilon_2}\frac{\tilde{\varepsilon}^2}{k}$$

Here: minor differences w.r.t. k-ε model:

- production is exact: $\mathcal{P} = -\langle u_i' u_j' \rangle \langle u_i \rangle_{,j}$
- turbulent diffusion term has anisotropic eddy viscosity: $(\nu_T)_{kl} = C_\varepsilon \frac{k}{\varepsilon}\langle u_k' u_l' \rangle$ with: $C_\varepsilon / \sigma_\varepsilon = 0.15$
- other constants take standard values: $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$

Predictions for mixing layer flow

Self-similar mixing layer

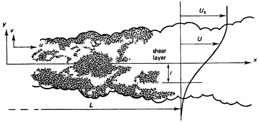
- spreading rate $\mathrm{d}\delta/\mathrm{d}x$:

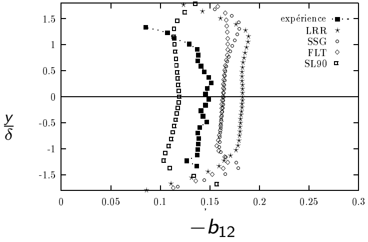
exp.	LRR	SSG	k-ε
0.019	0.019	0.018	0.016

$\longrightarrow$ LRR, SSG yield good predictions

- turbulence structure similar to homogeneous shear flow

$\Rightarrow$ performance as calibrated

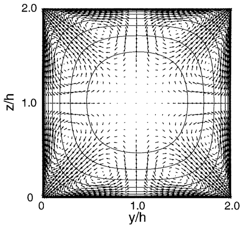


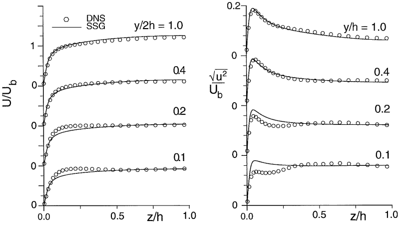


(experiment of Bell & Mehta, 1990)

Flow through a straight square duct

secondary flow vectors



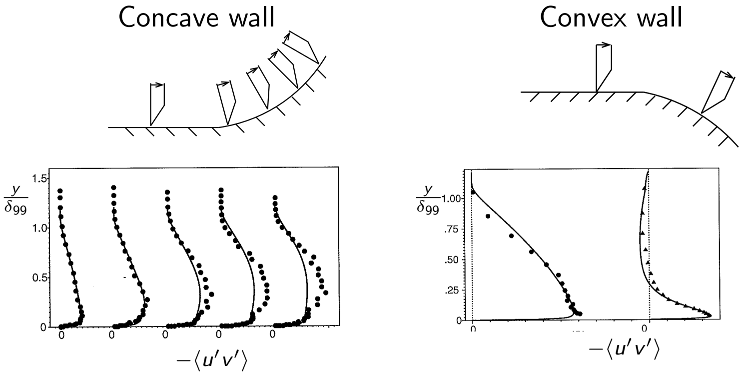


(from Durbin and Pettersson Reif, 2001)

$$\langle v\rangle \partial_y \langle \omega_x\rangle + \langle w\rangle \partial_z \langle \omega_x\rangle = \partial_{yz}\left(\langle v'v'\rangle - \langle w'w'\rangle\right) + \left(\partial_{yy} - \partial_{zz}\right)\langle v'w'\rangle + \nu\left(\partial_{yy} + \partial_{zz}\right)\langle \omega_x\rangle$$

- Reynolds-stress model predicts: secondary shear stress $\langle v'w'\rangle$ & normal-stress anisotropy $\langle v'v'\rangle - \langle w'w'\rangle$
- here: secondary flow strength underpredicted

Boundary layer on curved walls



• experiments, —— Reynolds-stress model (Durbin 1993)

- increased production

$\longrightarrow$ enhanced turbulence

- reduced production

$\longrightarrow$ damped turbulence

$\Rightarrow$ captured by Reynolds stress transport model

Summary

Why resort to Reynolds-stress models?

- convective transport and production mechanisms are exact

$\langle u_i' u_j' \rangle$ transport equation derived from Navier-Stokes

Pressure-strain correlation is principal unknown

- splitting in slow and rapid part
- modelling for homogeneous flow (tensor fct., realizability)

Performance of Reynolds-stress models:

- account for complex straining fields, normal stress anisotropy

Outlook: Boundary conditions and wall treatment

How can RANS models be applied in wall-bounded flows?

- the wall-function approach
- specific model modifications for the wall region

Further reading

- S. Pope, *Turbulent flows*, 2000
  $\longrightarrow$ chapter 11
- P.A. Durbin and B.A. Pettersson Reif,
  Statistical theory and modeling for turbulent flows, 2003
  $\longrightarrow$ chapter 7